A geometrical interpretation of black hole space-time in 2+1 dimensions

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Abstract

A two-dimensional surface representing a black hole space-time is generated by using numerical techniques. This is achieved by comparing the motion of a massive particle on the black hole space-time to a particle on a surface of revolution in Newtonian gravity.

Key words: surface of revolution, cosmological constant, black holes

Introduction

The general theory of relativity is the theory of gravitation developed by Einstein in 1915 (Wald). In the first few decades after its origin, the general theory of relativity was not extended due to its mathematical and conceptual complexities. However, experiments leading to the discovery of astronomical objects such as quasars, pulsars, black holes, and cosmic background radiation have placed it at the forefront of research. Since gravity is the dominant force in the large scale structure of the universe, general relativity has played an important role in cosmology. For example, gravitational lensing, which is a very actively pursued means of obtaining information on large scale mass distributions of the universe, has its roots in the general theory of relativity.

One of the most profound predictions of general relativity is the existence of black holes (Chandrasekhar). The name “black hole” is derived from the phenomenon’s conceptualization as a region of space from which not even light can escape. Since they are not directly observable, one can only infer the existence of black holes by collecting astronomical data. Recent discoveries show that giant black holes exist in the center of some galaxies (Rees). Hence, studies of black hole physics take an important place in understanding the structure of our universe. The mathematical complexity of general relativity, however, often obstructs the visualization of the physics behind black holes. Consequently, it is interesting to construct a model in the lab to help understand the behavior of particles in a black hole space-time. The purpose of this paper is to describe how a surface of revolution with appropriate geometry would act as a “toy model” for a black hole space-time in 2+1 dimensions.

This paper is organized as follows. We will first describe the motion of a particle on a surface of revolution in Newtonian gravity. Subsequently, we will provide an introduction to the black hole in 2+1 dimensions and the geodesics of a particle. Next, we will discuss a mechanical model for a particle moving around a black hole. Finally, we will suggest the implications of this research.

Motion of a Particle on a Surface of Revolution

A mechanical model can be constructed for a particle describing a circular orbit on a surface of revolution around the Z-axis. A popular model of such a construction is one in science museums with a surface of revolution . The curvature of the surface combined with gravity leads to a horizontal acceleration of for a particle in a circular orbit.

In a recent paper (McDonald), circular orbits of a particle on a general surface of revolution were studied. It was revealed that such a surface could have a critical radius where the circular orbits are unstable and fall off the surface. Given the fact that such a surface is two dimensional and the time slices of a 2+1 dimensional black hole are also two dimensional, one could argue that the circular orbits around a black hole
can be modeled in a lab with such a surface of revolution. Therefore, in this section, we will first describe the characteristics of a particle on a surface of revolution.

Let an arbitrary surface of revolution be given by \( z = f(r) \). The motion of a particle can be described in a cylindrical coordinate system \((r, \theta, z)\), with the Z-axis vertical. Assuming there is no friction on the surface, the angular momentum, \( J = r^2 \dot{\theta} \) of the particle around the Z-axis would be constant. The kinetic and the gravitational potential energy of a particle with unit mass is given by

\[
T = \frac{1}{2} \left( r^2 + r^2 \dot{\theta}^2 + z^2 \right) \tag{1}
\]

and

\[
V = g z \tag{2}
\]

Hence the total energy is

\[
E_1 = T + V = \frac{1}{2} \left( r^2 + \frac{J^2}{r^2} + (f')^2 \right) + gf \tag{3}
\]

Here, the dot represents the derivative with respect to \( t \) and the prime represents the derivative with respect to the \( r \) coordinate. Since there is no friction, the total energy will be conserved. The above equation can be rearranged to be of the form

\[
r^2 + \frac{1}{(1 + (f')^2)} \left( \frac{J^2}{r^2} + 2gf - 2E_1 \right) = 0 \tag{4}
\]

Using perturbation techniques, McDonald showed that for unstable circular orbits to exist, the function \( r = f^{-1}(z) \) should satisfy the condition \((1/r^2)^r < 0\).

**Black Hole in 2+1 Dimensions**

Banados et al obtained black hole solutions in 2+1 dimensions with a negative cosmological constant. Since this black hole (named BTZ black hole) is technically simpler than 3+1 dimensions, it has attracted great interest. This solution is obtained by solving Einstein’s field equations with a negative cosmological constant given by

\[
R_{\mu \nu} - g_{\mu \nu} (R - \Lambda) = 8 \pi G T_{\mu \nu} \tag{5}
\]

Here, \( \Lambda = -1/r^2 \) and \( 8 \pi G = 1 \). It is assumed that the space-time is stationary and circularly symmetrical. Solving Eq. 5 with \( T_{\mu \nu} = 0 \) leads to the BTZ black hole given by the following metric:

\[
ds^2 = - \left( \frac{r^2}{l^2} - M + \frac{a^2}{4r^2} \right) dt^2 + \frac{dr^2}{\left( \frac{r^2}{l^2} - M + \frac{a^2}{4r^2} \right)} + r^2 \left( d\theta - \frac{a}{2r^2} dt \right)^2 \tag{6}
\]

Here, \( M \) represents the mass and \( a \) represents angular momentum of the black hole. For \( M > 0 \) and \( |a| < Ml \), the above metric has horizons at

\[
r^2 = \frac{ML^2}{2} \left[ 1 \pm \sqrt{1 - \left( \frac{a}{Ml} \right)^2} \right] \tag{7}
\]

The main purpose of this paper is to create a surface of revolution that describes the motion of a massive particle in the vicinity of a 2+1 dimensional black hole. Such a motion is described by time-like geodesics of the black hole space-time. Note that in this paper we consider only a spin less black hole with \( a = 0 \). Equations governing the geodesics in a space-time with the line element

\[
ds^2 = g_{ab} dx^a dx^b \tag{8}
\]

can be derived from the Lagrangian

\[
2L = g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \tag{9}
\]

where \( \tau \) is some affine parameter along the geodesic. For time-like geodesics, \( \tau \) may be identified with the proper time measured by an observer traveling along...
the geodesics. The geodesics can be derived from $L$ as follows: Since the notion of length is described by the metric, the length between two space-time points $S$ and $T$ are given by

$$\text{Length} = \int_{S}^{T} \sqrt{-ds^2} = \int_{S}^{T} \sqrt{-g_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}}$$  \hspace{1cm} (10)$$

This could also be written in terms of the Lagrangian as

$$\text{Length} = \int_{S}^{T} \sqrt{2L} d\tau$$  \hspace{1cm} (11)$$

Considering the fact that a geodesic is a curve of extreme length between two points in space-time, extremizing Eq. 11 leads to an equation of the form

$$\frac{\partial L}{\partial x^{\mu}} \frac{d}{d\tau} \left( \frac{\partial L}{\partial (\partial x^{\mu})} \right) = 0$$  \hspace{1cm} (12)$$

Here, $x^{\mu}$ corresponds to one of $(t, r, \theta)$ in 2+1 dimensions. For the BTZ black hole, the Lagrangian $L$ is given by

$$L = \frac{1}{2} \left[ -N(r) \left( \frac{dt}{d\tau} \right)^2 + \frac{1}{N(r)} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\theta}{d\tau} \right)^2 \right]$$  \hspace{1cm} (13)$$

Here, $N(r) = (r^2/l^2 - M)$. For $x^{\mu} = \theta$, Eq. 12 becomes

$$\frac{\partial L}{\partial \theta} \frac{d}{d\tau} \left( \frac{\partial L}{\partial (\partial \theta)} \right) = 0$$  \hspace{1cm} (14)$$

which leads to

$$\frac{\partial L}{\partial \theta} = \text{constant} \Rightarrow r^2 \dot{\theta} = J_2$$  \hspace{1cm} (15)$$

For $x^{\mu} = t$, Eq. 12 becomes

$$\frac{\partial L}{\partial t} \frac{d}{d\tau} \left( \frac{\partial L}{\partial (\partial t)} \right) = 0$$  \hspace{1cm} (16)$$

which leads to

$$\frac{\partial L}{\partial t} = \text{constant} \Rightarrow \left( \frac{r^2}{l^2} - M \right) \frac{\dot{r}}{r} = E_2$$  \hspace{1cm} (17)$$

Note that the constants of motion in the black hole setting is given by $J_2$ and $E_2$, while the constants of motion in the Newtonian setting described in section 2 is given by $J_1$ and $E_1$. Furthermore, it is worthwhile to note that the constant $E_2$ for geodesics cannot be interpreted as the local energy of a particle at infinity since the black hole space-time is not asymptotically flat.

For time-like geodesics $L = -1$, combining this with Eq. 15 and Eq. 17 leads to

$$-1 = \frac{1}{2} \left[ -\frac{E_2^2}{N(r)} + \frac{J_2}{N(r)} + \frac{J_2^2}{r^2} \right]$$  \hspace{1cm} (18)$$

The above equation can be rewritten as

$$\frac{J_2^2}{2} + \frac{1}{2} \left( \frac{r^2}{l^2} - M \right) \left( \frac{J_2^2}{r^2} + 2 \right) = \frac{E_2^2}{2}$$  \hspace{1cm} (19)$$

This equation shows that the radial motion of a geodesic is represented by the same equation of motion as a unit mass particle with energy $(E_2^2)/2$ in ordinary one-dimensional non-relativistic mechanics with the effective potential

$$V_{eff} = \frac{1}{2} \left( \frac{r^2}{l^2} - M \right) \left( \frac{J_2^2}{r^2} + 2 \right)$$  \hspace{1cm} (20)$$

From Eq. 19, it is clear that there will always be a finite upper bound $r_{max}$ for the radial coordinate for time-like geodesics. Hence, the massive particles cannot escape from the black hole. We describe this as follows: Since $\dot{r}$ has to be real, the expression

$$E_2^2 - \left( \frac{r_0^2}{l^2} - M \right) \left( \frac{J_2^2}{r_0^2} + 2 \right) \geq 0$$  \hspace{1cm} (21)$$

which leads to $r \geq r_{max}$. Here $r_{max}$ is given by
From the above arguments, it is clear that a time-like geodesic that starts at a point \( r > r_{\text{max}} \) can cross the horizon at \( r = M \) and hit the singularity at \( r = 0 \). The behavior of time-like geodesics for this black hole is discussed in detail in the work by Cruz et al.

In Figure 1, the effective potential for a particle with zero angular momentum \( (J_2 = 0) \) is given. We have picked the values \( M = l = 1 \). One can see that a particle released from a distance will fall into the black hole. In Figure 2, the effective potential for a particle with angular momentum \( (J_2 = 1) \) is given. Here, \( M = l = 1 \). From the shape of the effective potential it is clear that there are no stable circular orbits. In fact this is true for all values of \( M > 0 \) and \( l \) for non-spinning black holes.

**Mechanical model for the black hole**

In this section we compare the motion of a massive particle in the BTZ black hole space-time and a particle moving on a surface of revolution given by Eq. 4 and Eq. 19. This will lead to the function \( f(r) \) given by the following differential equation:

\[
\frac{P(r)}{(1 + (r')^2)} = Q(r)
\]  

(24)

where

\[
Q(r) = \frac{1}{2} \left( \frac{r^2}{r^2 - 1} - \frac{J_2^2}{r^2} + 2 M \right)
\]  

(25)

\[
P(r) = \frac{1}{2} \left( \frac{J_1^2}{r^2} + 2 g f(r) - 2 E_1 \right)
\]  

(26)

This leads to the final differential equation for \( f(r) \) as

\[
\left( \frac{df(r)}{dr} \right)^2 = \frac{P(r)}{Q(r)} - 1
\]  

(27)

For various values of the parameters \( M, l, J_1, J_2, E_1, \) and \( E_2 \), the solution for \( f(r) \) would give different surfaces of revolution. Note that from Eq. 4 and Eq. 19, the two functions \( P(r) \) and \( Q(r) \) must be negative. On the other hand, from Eq. 27 it is obvious that \( P(r)/Q(r) \geq 1 \).

**Model for a radially falling particle**

The differential equation appearing in the above equation has to be solved numerically. First, we will solve it for a particle with zero angular momentum with the values \( J_1 = J_2 = 0 \). We will also set \( 2M = 2l^2 = E_1 = E_2 = g = 1 \). For these parameters, the function \( Q(r) \) is given by
\[ Q(r) = r^2 - \frac{3}{2} \]  

(28)

Since \( Q(r) \leq 0 \), the range for \( r \) considered is restricted to \( 0 \leq r \leq 1.225 \). For this case,

\[ P(r) = f(r) - 1 \]  

(29)

Since \( P(r) \leq 0 \), the function \( f(r) \leq 1 \). Also,

\[ \frac{P(r)}{Q(r)} \geq 1 \]  

leads to the condition \( f(r) \leq r^2 - \frac{1}{2} \). We find solutions to \( f(r) \) with these restrictions with the boundary value \( f(0) = -0.8 \). Eq. 24 is solved for the range \( 0 \leq r \leq 1.22 \). Eq. 24 is solved for the range \( 0 \leq r \leq 1.22 \), which is given in Figure 3. The corresponding surface of revolution is given in Figure 4.

Figure 3. The function \( f(r) \) for the parameters \( M=1, l=1, J_1=J_2=0 \).

Figure 4. The surface of revolution for a particle falling into a black hole.
Model for a particle with angular momentum

We pick the values $J_1 = J_2 = 1$ to describe a particle with angular momentum. We will also set $2M = 2l^2 = E_1 = g = E_2 = 1$. For these parameters, the function $Q(r)$ is given by

$$Q(r) = r^2 - \frac{1}{2r^2} - 1$$

Since $Q(r) \leq 0$, $r$ is restricted to the range $0 \leq r \leq 1.169$ for the motion considered. For this case, $P(r)$ is

$$P(r) = \frac{1}{2} \left( 2f(r) - 2 + \frac{1}{r^2} \right)$$

Since $P(r) \leq 0$, the function $f(r) \leq (1 - 1/2r^2)$. Furthermore, for $P(r)/Q(r) \geq 1$ gives $f(r) \leq (r^2 - 1/r^2)$. In the range for $0 \leq r \leq 1.169$, $(r^2 - 1/r^2) \leq (1 - 1/2r^2)$.

Hence the restriction for $f(r)$ becomes $f(r) \leq (r^2 - 1/r^2)$. We find solutions to $f(r)$ with these restrictions with the boundary value $f(0.3) = -12$. Eq. 24 is solved for the range $0.3 \leq r \leq 1.16$ which is given in Figure 5. The surface of revolution corresponding to that is given in Figure 6.
Conclusion
In conclusion, we have derived the function for a surface of revolution to describe the behavior of a particle moving around a black hole in 2+1 dimensions. Even though the function cannot be given in closed form, one can perform numerical integration for suitable values of the parameters to visualize the surfaces. It would be interesting to extend this work to the rotating black hole.

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References