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Chapter 1 Section 1:

Chapter 1. Algebraic Expressions

1.1 Integer Exponents

The product of several copies of the same number is generally written in *exponential notation*. For example, the symbol $5^3 = 5 \cdot 5 \cdot 5$ represents the product of 3 copies of 5 multiplied together. We generalize this statement in Definition 1.

Definition 1. If a is a real number and n is a positive integer, then a to the power n is

$$a^n = a \cdot a \cdot \ldots \cdot a \qquad (n \text{ factors of } a)$$

The number a is called the *base* and the number n is called the *exponent*.

The properties of exponents can be easily discovered using simple examples. For instance, notice that

$$5^3 \cdot 5^4 = (5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5) = 5^7$$

We generalize this result for any base a and positive integer exponents m and n to discover the property $a^m \cdot a^n = a^{m+n}$. In words, if we multiply m factors of a by n factors of a we obtain $m+n$ factors of a.

We would like this property to be true even when m and n are 0 or negative integers. For example, if this property is true, then we must have, for any $a \neq 0$,

$$a^0 \cdot a^3 = a^{0+3} = a^3$$

If we divide by a^3, we obtain $a^0 = 1$.

Similarly, if n is a positive integer and this property is true, then we must have, for any $a \neq 0$,

$$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1$$

If we divide by a^n, we obtain $a^{-n} = \frac{1}{a^n}$. If we divide by a^{-n}, we obtain $a^n = \frac{1}{a^{-n}}$.

These observations lead to the following definition.

Definition 2. If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n} \qquad \text{and} \qquad a^n = \frac{1}{a^{-n}}$$

With this definition, we are able to state five properties of exponents that apply whenever the base is any nonzero real number and the exponent is any integer, positive, negative or zero. These are shown in the table below. Some examples of how each property is applied are given.

Properties of Exponents

If $a \neq 0$ and $b \neq 0$ are real numbers, m and n are integers, then

Property	Examples	
1. $a^m a^n = a^{m+n}$	$5^3 \cdot 5^4 = 5^{3+4} = 5^7$	$5^2 \cdot 5^{-6} = 5^{2-6} = 5^{-4} = \dfrac{1}{5^4}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{3^7}{3^2} = 3^{7-2} = 3^5$	$\dfrac{3^{-8}}{3^{-2}} = 3^{-8-(-2)} = 3^{-6} = \dfrac{1}{3^6}$
3. $(a^m)^n = a^{mn}$	$(7^2)^3 = 7^{2\cdot 3} = 7^6$	$(7^3)^{-2} = 7^{3(-2)} = 7^{-6} = \dfrac{1}{7^6}$
4. $(ab)^n = a^n b^n$	$(3\cdot 5)^2 = 3^2 \cdot 5^2$	$(3\cdot 5)^{-2} = 3^{-2}\cdot 5^{-2} = \dfrac{1}{3^2}\cdot\dfrac{1}{5^2} = \dfrac{1}{3^2\cdot 5^2}$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{5}{3}\right)^4 = \dfrac{5^4}{3^4}$	$\left(\dfrac{5}{3}\right)^{-4} = \dfrac{5^{-4}}{3^{-4}} = \dfrac{1/5^4}{1/3^4} = \dfrac{1}{5^4}\cdot\dfrac{3^4}{1} = \dfrac{3^4}{5^4}$

Positive Integer Exponents

We can use the properties of exponents to simplify algebraic expressions involving positive exponents as discussed in Example 1.

Example 1. Simplify the given expression, writing it without any negative exponents.

(a) $\dfrac{x^3}{x^5}$ **(b)** $(5u^3v^4)^2$ **(c)** $(3a^2b)(-2a^3b^4)^3$ **(d)** $\left(\dfrac{x^3}{4y}\right)^2\left(\dfrac{x^4y^2}{z}\right)^3$

Solution. **(a)** We can apply Property 2 and Definition 2 to obtain

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2} \qquad \text{Property 2: } \frac{a^m}{a^n} = a^{m-n}$$

$$= \frac{1}{x^2} \qquad \text{Definition 2: } a^{-n} = \frac{1}{a^n}$$

In this example, it is easier to use Property 1 and cancel to obtain

$$\frac{x^3}{x^5} = \frac{x^3}{x^3x^2} \qquad \text{Property 1: } a^m a^n = a^{m+n}$$

$$= \frac{1}{x^2} \qquad \text{Divide out (cancel) } x^3$$

(b) We use Property 4 and Property 3 to obtain

$$(5u^3v^4)^2 = 5^2(u^3)^2(v^4)^2 \qquad \text{Property 4: } (ab)^n = a^n b^n$$

$$= 25u^6v^8 \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

(c) We use Property 4, Property 3 and Property 1 to obtain

$$(3a^2b)(-2a^3b^4)^3 = (3a^2b)(-2)^3(a^3)^3(b^4)^3 \qquad \text{Property 4: } (ab)^n = a^n b^n$$

$$= (3a^2b)(-8)a^9b^{12} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= -24a^{11}b^{13} \qquad \text{Property 1: } a^m a^n = a^{m+n}$$

(d) We use all five properties to obtain

$$\left(\frac{x^3}{4y}\right)^2\left(\frac{x^4y^2}{z}\right)^3 = \frac{(x^3)^2}{(4y)^2}\cdot\frac{(x^4y^2)^3}{z^3} \qquad \text{Property 5: } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$= \frac{x^6}{4^2y^2}\cdot\frac{(x^4y^2)^3}{z^3} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= \frac{x^6}{4^2y^2}\cdot\frac{(x^4)^3(y^2)^3}{z^3} \qquad \text{Property 4: } (ab)^n = a^n b^n$$

$$= \frac{x^6x^{12}y^6}{16y^2z^3} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= \frac{x^{18}y^6}{16y^2z^3} \qquad \text{Property 1: } a^m a^n = a^{m+n}$$

$$= \frac{x^{18}y^4}{16z^3} \qquad \text{Property 2: } \frac{a^m}{a^n} = a^{m-n}$$

Negative Integer Exponents

We can also use the properties of exponents to simplify algebraic expressions that involve both positive and negative integer exponents. There is a strategy that can be used to simplify this process and help reduce mistakes. It involves converting an expression involving negative exponents to equivalent expression involving only positive exponents. We explain this strategy in the following examples.

Example 2. Convert the expression $\dfrac{x^{-3}y^{-2}}{x^5y^{-7}z^{-4}}$ to an equivalent expression involving only positive exponents.

Solution. We write the expression as the product of single factors, apply Definition 2, and rewrite as a single expression with all positive exponents. We obtain

$\dfrac{x^{-3}y^{-2}}{x^5y^{-7}z^{-4}} = x^{-3}\cdot\dfrac{1}{x^5}\cdot y^{-2}\cdot\dfrac{1}{y^{-7}}\cdot\dfrac{1}{z^{-4}}$	Multiplication of fractions
$= \dfrac{1}{x^3}\cdot\dfrac{1}{x^5}\cdot\dfrac{1}{y^2}\cdot y^7\cdot z^4$	Definition 2: $a^{-n}=\dfrac{1}{a^n}$; $a^n=\dfrac{1}{a^{-n}}$
$= \dfrac{y^7z^4}{x^3x^5y^2}$	Multiplication of fractions

We do not include this detailed explanation in the examples that follow. Instead, we simply write

$\dfrac{x^{-3}y^{-2}}{x^5y^{-7}z^{-4}} = \dfrac{y^7z^4}{x^3x^5y^2}$	Convert to positive exponents

Example 3. Simplify the given expression, writing it without any negative exponents.

(a) $(-3x^2y^{-3})^2(x^{-3}y^2)^{-5}$ **(b)** $\left(\dfrac{a^2b}{c^3}\right)^{-2}$ **(c)** $\left(\dfrac{5x^{-4}y^3}{2x^2y^{-1}}\right)^{-3}$

Solution. **(a)** We use Property 4, Property 3, Property 1 and convert to positive exponents to obtain

$(-3x^2y^{-3})^2(x^{-3}y^2)^{-5} = (-3)^2(x^2)^2(y^{-3})^2(x^{-3})^{-5}(y^2)^{-5}$	Property 4: $(ab)^n=a^nb^n$
$= 9x^4y^{-6}x^{15}y^{-10}$	Property 3: $(a^m)^n=a^{mn}$
$= 9x^{19}y^{-16}$	Property 1: $a^ma^n=a^{m+n}$
$= \dfrac{9x^{19}}{y^{16}}$	Convert to positive exponent

(b) We use Property 5, Property 4, and convert to positive exponents to obtain

$\left(\dfrac{a^2b}{c^3}\right)^{-2} = \dfrac{(a^2b)^{-2}}{(c^3)^{-2}}$	Property 5: $\left(\dfrac{a}{b}\right)^n=\dfrac{a^n}{b^n}$
$= \dfrac{a^{-4}b^{-2}}{c^{-6}}$	Property 4: $(ab)^n=a^nb^n$

$$= \frac{c^6}{a^4 b^2}$$ Convert to positive exponents

(c) We use Property 5, Property 4, convert to positive exponents, and then use Property 1 to obtain

$$\left(\frac{5x^{-4}y^3}{2x^2y^{-1}}\right)^{-3} = \frac{(5x^{-4}y^3)^{-3}}{(2x^2y^{-1})^{-3}}$$ Property 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$= \frac{5^{-3}x^{12}y^{-9}}{2^{-3}x^{-6}y^3}$$ Property 4: $(ab)^n = a^n b^n$

$$= \frac{2^3 x^{12} x^6}{5^3 y^3 y^9}$$ Convert to positive exponents

$$= \frac{8x^{18}}{125y^{12}}$$ Property 1: $a^m a^n = a^{m+n}$

Example 4. Find the value of the algebraic expression $\frac{x^{-3}y}{x^{-2}y^{-2}}$ when $x = -3$ and $y = -2$.

Solution. It is a good idea to simplify this complicated expression before evaluating it. We first convert to all positive exponents, simplify using
Property 1 and canceling to obtain

$$\frac{x^{-3}y}{x^{-2}y^{-2}} = \frac{x^2 y y^2}{x^3}$$ Convert to positive exponents

$$= \frac{x^2y^3}{x^3}$$ Property 1: $a^m a^n = a^{m+n}$

$$= \frac{x^2y^3}{x^2x}$$ Write $x^3 = x^2 x$

$$= \frac{y^3}{x}$$ Divide out (cancel) x^2

It is now a simple matter to find the value of the simplified expression $\frac{y^3}{x}$ at $x = -3$ and $y = -2$ to obtain

$$\frac{(-2)^3}{-3} = \frac{-8}{-3} = \frac{8}{3}$$

Scientific Notation

Scientists use exponential notation to write, multiply and divide very large or very small numbers. Such numbers are difficult to read, write, multiply and divide, so scientists generally express them in *scientific notation*.

Definition 3. A positive number x is in scientific notation if it is written $x = a \times 10^n$, where $1 \leq a < 10$ is a real number and n is an integer.

Example 5. Write the following numbers in scientific notation.

a) $35,780,000$ **b)** 0.00001492

Solution. **(a)** Take the decimal number 3.578 and move the decimal point 7 places to the right by multiplying it by 10^7 to obtain $3.578 \times 10^7 = 35,780,000$.

(b) Take the decimal number 1.492 and move the decimal point 5 places to the left by multiplying it by 10^{-5} to obtain $1.492 \times 10^{-5} = 0.00001492$.

Example 6. Suppose $a = 0.0054$, $b = 8,300,000,000$ and $c = 1.7 \times 10^{25}$. Write these numbers in scientific notation and perform the indicated operations.

(a) ab **(b)** $\dfrac{bc}{a}$

Solution. **(a)** We first write a and b in scientific notation: $a = 5.4 \times 10^{-3}$ and $b = 8.3 \times 10^9$. With the aid of a calculator, we obtain

$$\begin{aligned} ab &= (5.4 \times 10^{-3}) \cdot (8.3 \times 10^9) = (5.4 \times 8.3) \cdot (10^{-3} \times 10^9) \\ &= 44.82 \times 10^6 = 4.482 \times 10 \times 10^6 \\ &= 4.482 \times 10^7 \end{aligned}$$

(b) We have

$$\frac{bc}{a} = \frac{(8.3 \times 10^9) \cdot (1.7 \times 10^{25})}{5.4 \times 10^{-3}} = \frac{8.3 \times 1.7}{5.4} \times 10^{9+25-(-3)} = 2.61296 \times 10^{37}$$

1.2 Roots and Rational Exponents

Roots

We begin with two definitions of the nth root of a real number, one when n is a positive even integer and the other when n is an odd positive integer.

Definition 1. If n is an even positive integer, the nthroot of a nonnegative real number a is that unique nonnegative real number, denoted by $\sqrt[n]{a}$, such that $\left(\sqrt[n]{a}\right)^n = a$. When $n = 2$, we write $\sqrt[2]{a}$ as $\sqrt{a}$, the square root of a.

It follows that $\sqrt[4]{81} = 3$ since $3^4 = 81$, and that $\sqrt[6]{64} = 2$ since $2^6 = 64$. On the other hand, notice that $\sqrt{-4}$is not defined since there is no real number whose square is -4. This is true since the square of any real number is always nonnegative. Similarly, $\sqrt[4]{-81}$ is not defined since there is no real number whose fourth power is -81. *Thus when n is even, the nth root of a negative number is not defined for this same reason*.

Definition 2. If n is an odd positive integer greater than 1, the nth root of a real number a is that unique real number, denoted by $\sqrt[n]{a}$, such that $\left(\sqrt[n]{a}\right)^n = a$.

Note that when n is odd, the nth root is defined for *any* real number. We have, for example, $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$ and $\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$.

Rational Exponents

To define what we mean by a *rational exponent*, we begin with rational number exponents of the form $1/n$ where $n > 1$ is a positive integer.

Definition 3. For any positive integer $n > 1$, we define $a^{1/n}$, for a a real number, as $a^{1/n} = \sqrt[n]{a}$. If n is even, we must have $a \geq 0$.

We can then extend this definition to all rational exponents.

Definition 4. For any rational number m/n in lowest terms, where m and n are integers and $n > 1$, we define

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \quad \textit{or equivalently} \quad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, we must have $a \geq 0$.

With these definitions, it can be shown that The Properties of Exponents for *integer* exponents we considered in Section 1.1 also hold for *rational* exponents.

Properties of Rational Exponents

If p and q are rational numbers in lowest terms with positive integer denominators, then

Property	**Examples**	
1. $a^p a^q = a^{p+q}$	$5^{3/2}5^{1/2} = 5^{3/2+1/2} = 5^2 = 25$	$8^{-1/3}8^{2/3} = 8^{-1/3+2/3} = 8^{1/3} = 2$
2. $\dfrac{a^p}{a^q} = a^{p-q}$	$\dfrac{4^{7/2}}{4^{3/2}} = 4^{7/2-3/2} = 4^2 = 16$	$\dfrac{7^{5/3}}{7} = 7^{5/3-1} = 7^{2/3} = \sqrt[3]{7^2}$
3. $\left(a^p\right)^q = a^{pq}$	$\left(2^{3/4}\right)^4 = 2^{3/4\cdot 4} = 2^3 = 8$	$(8)^{-2/3} = \left((8)^{1/3}\right)^{-2} = (-2)^{-2} = \dfrac{1}{4}$
4. $(ab)^p = a^p b^p$	$(16\cdot 3)^{1/2} = 16^{1/2}\cdot 3^{1/2} = 4\sqrt{3}$	$(8\cdot 4)^{1/3} = 8^{1/3}\cdot 4^{1/3} = 2\sqrt[3]{4}$
5. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$	$\left(\dfrac{16}{9}\right)^{1/2} = \dfrac{16^{1/2}}{9^{1/2}} = \dfrac{4}{3}$	$\left(\dfrac{8}{27}\right)^{2/3} = \dfrac{8^{2/3}}{27^{2/3}} = \dfrac{(8^{1/3})^2}{(27^{1/3})^2} = \dfrac{4}{9}$

Each property holds for real numbers a and b for which the expressions on both sides of the equality are defined.

Two useful properties of the nth root follow from Property 4 and Property 5.

Product Property. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ *Proof*: $\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a}\sqrt[n]{b}$

Quotient Property. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ *Proof*: $\sqrt[n]{\dfrac{a}{b}} = \left(\dfrac{a}{b}\right)^{1/n} = \dfrac{a^{1/n}}{b^{1/n}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Rational exponents and their properties provide important ways to do calculations. The square root of a number can be calculated using the square root key on any calculator. However, if we want to compute the cube root or any other root of a number, or any rational power of a number, we must use rational exponents. For example, if we want to calculate the value of $\sqrt[3]{2}$, we must enter this number as $2\wedge(1/3)$ since there is no cube root key on a calculator.

Example 1 illustrates how expressions and numbers in radical form can be expressed in exponential form and vice versa.

Example 1. Write each expression in radical form as an expression in exponential form and each expression in exponential form as an expression in radical form.

(a) $\sqrt[5]{x^2}$ **(b)** $\dfrac{1}{\sqrt{x^3}}$ **(c)** $3^{4/7}$ **(d)** $5^{-2/3}$

Solution. **(a)** We write

$$\sqrt[5]{x^2} = \left(x^2\right)^{1/5} \qquad \text{Convert to exponential form}$$

$$= x^{2/5} \qquad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

Note: We could also use Definition 4 to write $\sqrt[5]{x^2} = x^{2/5}$directly.

(b) We write

$$\frac{1}{\sqrt{x^3}} = \frac{1}{(x^3)^{1/2}} \qquad \text{Convert to exponential form}$$

$$= \frac{1}{x^{3/2}} \qquad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

(c) We write

$$3^{4/7} = (3^4)^{1/7} \qquad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

$$= \sqrt[7]{3^4} \qquad \text{Convert to radical form}$$

(d) We write

$$5^{-2/3} = \frac{1}{5^{2/3}} \qquad \text{Property 2*: } \frac{a^p}{a^q} = a^{p-q}$$

$$= \frac{1}{(5^2)^{1/3}} \qquad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

$$= \frac{1}{\sqrt[3]{5^2}} \qquad \text{Convert to radical form}$$

***Note.** The expression $\frac{a^p}{a^q} = a^{p-q}$ in Property 2 gives $\frac{1}{a^q} = a^{-q}$ when $p = 0$. Hence $5^{-2/3} = \frac{1}{5^{2/3}}$.

The following examples show how roots, rational exponents and the properties of exponents can be applied to evaluate numerical expressions and simplify algebraic expressions.

Example 2. Evaluate the given expression.

(a) $\sqrt{7}\sqrt{28}$ **(b)** $\sqrt[5]{\frac{1}{32}}$ **(c)** $9^{-3/2}$ **(d)** $\left(-\frac{8}{27}\right)^{2/3}$

***Solution*. (a)** We write

$$\begin{aligned}\sqrt{7}\sqrt{28} &= \sqrt{7}\sqrt{7\cdot 2^2} && \text{Factor: } 28 = 7\cdot 2^2\\ &= \sqrt{7}\sqrt{7}\sqrt{2^2} && \text{Product Property of Roots}\\ &= 7\cdot 2 && \text{Definition of Square Root}\\ &= 14 && \text{Evaluate: } 7\cdot 2 = 14\end{aligned}$$

(b) We could first convert $\sqrt[5]{\dfrac{1}{32}}$ to exponential form and use the properties of exponents to evaluate it. Instead, we use a slightly shorter approach using only roots.

$$\begin{aligned}\sqrt[5]{\frac{1}{32}} &= \frac{\sqrt[5]{1}}{\sqrt[5]{32}} && \text{Quotient Property of Roots}\\ &= \frac{\sqrt[5]{1}}{\sqrt[5]{2^5}} && \text{Factor: } 32 = 2^5\\ &= \frac{1}{2} && \text{Definition of Fifth Root}\end{aligned}$$

(c) We use the properties of exponents to write

$$\begin{aligned}9^{-3/2} &= \frac{1}{9^{3/2}} && \text{Convert to positive exponent}\\ &= \frac{1}{(9^{1/2})^3} && \text{Property 3: } \left(a^p\right)^q = a^{pq}\\ &= \frac{1}{3^3} && \text{Simplify: } 9^{1/2} = \sqrt{9} = 3\\ &= \frac{1}{27} && \text{Evaluate: } 3^3 = 27\end{aligned}$$

Note. We could also use Property 3 to write $9^{3/2} = \left(9^3\right)^{1/2} = \left(729\right)^{1/2} = 27$ but most of us would need a calculator to find the square root of 729!

(d) We use the properties of exponents to write

$$\begin{aligned}\left(-\frac{8}{27}\right)^{2/3} &= \frac{(-8)^{2/3}}{27^{2/3}} && \text{Property 5: } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}\\ &= \frac{\left((-8)^{1/3}\right)^2}{\left(27^{1/3}\right)^2} && \text{Property 3: } \left(a^p\right)^q = a^{pq}\\ &= \frac{(-2)^2}{3^2} && \text{Simplify: } (-8)^{1/3} = -2,\ 27^{1/3} = 3\\ &= \frac{4}{9} && \text{Simplify: } (-2)^2 = 4,\ 3^2 = 9\end{aligned}$$

Example 3. Simplify the given numerical expression.

(a) $\sqrt{75}+\sqrt{48}-\sqrt{27}$ **(b)** $\left(\dfrac{27}{16}\right)^{-2/3}$

Solution. **(a)** We simplify each of the three square roots to obtain

$$\begin{aligned} \sqrt{75}+\sqrt{48}-\sqrt{27} &= \sqrt{25\cdot 3}+\sqrt{16\cdot 3}-\sqrt{9\cdot 3} && \text{Factor} \\ &= \sqrt{25}\sqrt{3}+\sqrt{16}\sqrt{3}-\sqrt{9}\sqrt{3} && \text{Product Property of Roots} \\ &= 5\sqrt{3}+4\sqrt{3}-3\sqrt{3} && \text{Evaluate Square Roots} \\ &= 6\sqrt{3} && \text{Simplify} \end{aligned}$$

(b) We use properties of rational exponents to obtain

$$\begin{aligned} \left(\frac{27}{16}\right)^{-2/3} &= \frac{27^{-2/3}}{16^{-2/3}} && \text{Property 5: } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p} \\ &= \frac{16^{2/3}}{27^{2/3}} && \text{Convert to positive exponents} \\ &= \frac{(2^4)^{2/3}}{(3^3)^{2/3}} && \text{Factor: } 16=2^4,\ 27=3^3 \\ &= \frac{2^{8/3}}{3^2} && \text{Property 3: } (a^p)^q = a^{pq} \\ &= \frac{2^{6/3}\cdot 2^{2/3}}{3^2} && \text{Property 1: } a^{p+q} = a^p a^q \\ &= \frac{4\cdot 2^{2/3}}{9} && \text{Simplify} \end{aligned}$$

Example 4. Express the product $\sqrt{147}\cdot\sqrt{80}$ in the form $A\sqrt{B}$ where A and B are integers and B is as small as possible.

Solution. We factor $147=49\cdot 3$ and $80=16\cdot 5$ to remove the largest perfect square in each case. We then use the product property of roots to obtain

$$\sqrt{147}\cdot\sqrt{80}=\sqrt{49\cdot 3}\cdot\sqrt{16\cdot 5}=7\sqrt{3}\cdot 4\sqrt{5}=28\sqrt{15}$$

Thus $A=28$ and $B=15$, and B is as small as possible since 15 has no perfect square as a factor.

Example 5. Simplify the given expression and write your answer in radical form. Assume that all variables represent positive numbers.

(a) $\sqrt{72x^{18}y^{29}z^{41}}$ **(b)** $\sqrt[3]{32x^6y^{17}}$ **(c)** $\sqrt[3]{\dfrac{81x^{13}}{y^{21}}}$

Solution. We solve these problems by first converting the given radical expression to exponential form, using the properties of rational exponents to simplify it, and converting the result back into radical form. The problems could equally well be solved using the Product and Quotient Properties of Roots.

(a) We obtain

$\sqrt{72x^{18}y^{29}z^{41}} = (72x^{18}y^{29}z^{41})^{1/2}$	Convert to exponential form
$= \left[(36x^{18}y^{28}z^{40})(2yz)\right]^{1/2}$	Factor out largest even power of x, y and z, largest perfect square factor of 72
$= (36x^{18}y^{28}z^{40})^{1/2}(2yz)^{1/2}$	Property 4: $(ab)^p = a^p b^p$
$= 6x^9y^{14}z^{20}(2yz)^{1/2}$	Property 4 and Property 3: $(a^p)^q = a^{pq}$

If we convert the answer to radical form, we obtain $6x^9y^{14}z^{20}\sqrt{2yz}$.

(b) We obtain

$\sqrt[3]{32x^6y^{17}} = (32x^6y^{17})^{1/3}$	Convert to exponential form
$= \left[(8x^6y^{15})(4y^2)\right]^{1/3}$	Factor out largest power of x and y that is a multiple of 3; largest perfect cube factor of 32
$= (8x^6y^{15})^{1/3}(4y^2)^{1/3}$	Property 4: $(ab)^p = a^p b^p$
$= 2x^2y^5(4y^2)^{1/3}$	Property 4 and Property 3: $(a^p)^q = a^{pq}$

If we convert the answer to radical form, we obtain $\sqrt[3]{32x^6y^{17}} = 2x^2y^5\sqrt[3]{4y^2}$.

(c) We obtain

$\sqrt[3]{\dfrac{81x^{13}}{y^{21}}} = \left(\dfrac{81x^{13}}{y^{21}}\right)^{1/3}$	Convert to exponential form
$= \dfrac{(81x^{13})^{1/3}}{(y^{21})^{1/3}}$	Property 5: $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$
$= \dfrac{\left[(27x^{12})(3x)\right]^{1/3}}{(y^{21})^{1/3}}$	Factor out the largest power of x that is a multiple of 3; largest perfect cube factor of 81.
$= \dfrac{(27x^{12})^{1/3}(3x)^{1/3}}{(y^{21})^{1/3}}$	Property 4: $(ab)^p = a^p b^p$
$= \dfrac{3x^4(3x)^{1/3}}{y^7}$	Property 4 and Property 3: $(a^p)^q = a^{pq}$

If we convert to radical form, we obtain $\sqrt[3]{\dfrac{81x^{13}}{y^{21}}} = \dfrac{3x^4\sqrt[3]{3x}}{y^7}$.

Example 6. Simplify the given expression and express all answers using positive exponents only. Assume that all variables represent positive numbers.

(a) $\dfrac{x^{2/3}y^{-2/5}}{x^{-1/2}y^{3}}$ **(b)** $\dfrac{(8x^{3}y^{-5})^{-2/3}}{2x^{1/2}y^{-3}}$ **(c)** $\left(\dfrac{2z^{-3}}{5w^{2/7}}\right)^{-1}$

Solution. **(a)** There are several approaches we could take, but we begin by converting to positive exponents. We obtain

$$\begin{aligned}
\frac{x^{2/3}y^{-2/5}}{x^{-1/2}y^{3}} &= \frac{x^{2/3}x^{1/2}}{y^{3}y^{2/5}} && \text{Convert to positive exponents}\\
&= \frac{x^{2/3+1/2}}{y^{3+2/5}} && \text{Property 1: } a^{p}a^{q}=a^{p+q}\\
&= \frac{x^{7/6}}{y^{17/5}} && \text{Add fractions in the exponents}
\end{aligned}$$

(b) We begin by simplifying the numerator. We obtain

$$\begin{aligned}
\frac{(8x^{3}y^{-5})^{-2/3}}{2x^{1/2}y^{-3}} &= \frac{8^{-2/3}(x^{3})^{-2/3}(y^{-5})^{-2/3}}{2x^{1/2}y^{-3}} && \text{Property 4: } (ab)^{p}=a^{p}b^{p}\\
&= \frac{8^{-2/3}x^{-2}y^{10/3}}{2x^{1/2}y^{-3}} && \text{Property 3: } (a^{p})^{q}=a^{pq}\\
&= \frac{y^{10/3}y^{3}}{8^{2/3}\cdot 2x^{1/2}x^{2}} && \text{Convert to positive exponents}\\
&= \frac{y^{10/3+3}}{4\cdot 2x^{1/2+2}} && \text{Property 1: } a^{p}a^{q}=a^{p+q},\\
& && 8^{2/3}=(8^{1/3})^{2}=4\\
&= \frac{y^{19/3}}{8x^{5/2}} && \text{Add fractions in the exponents}
\end{aligned}$$

(c) We obtain

$$\begin{aligned}
\left(\frac{2z^{-3}}{5w^{2/7}}\right)^{-1} &= \frac{(2z^{-3})^{-1}}{(5w^{2/7})^{-1}} && \text{Property 5: } \left(\frac{a}{b}\right)^{p}=\frac{a^{p}}{b^{p}}\\
&= \frac{2^{-1}z^{3}}{5^{-1}w^{-2/7}} && \text{Property 4: } (ab)^{p}=a^{p}b^{p}\\
&= \frac{5w^{2/7}z}{2} && \text{Convert to positive exponents}
\end{aligned}$$

Rationalizing the Denominator

It is common practice to write fractions like $\dfrac{1}{\sqrt{2}}$ as an equivalent faction with no radicals in the denominator by multiplying numerator and denominator by $\sqrt{2}$ to obtain

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Example 7 shows how this is done in a variety of other cases.

Example 7. Rationalize the denominator of the given fraction. Assume x is a positive number.

(a) $\dfrac{2}{7\sqrt{3}}$ **(b)** $\dfrac{1}{\sqrt[3]{x}}$ **(c)** $\sqrt[5]{\dfrac{4}{x^2}}$

***Solution*.** **(a)** $\dfrac{2}{7\sqrt{3}} = \dfrac{2}{7\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{21}$

(b) $\dfrac{1}{\sqrt[3]{x}} = \dfrac{1}{x^{1/3}} = \dfrac{1}{x^{1/3}} \cdot \dfrac{x^{2/3}}{x^{2/3}} = \dfrac{x^{2/3}}{x}$

(c) $\sqrt[5]{\dfrac{4}{x^2}} = \dfrac{4^{1/5}}{x^{2/5}} = \dfrac{4^{1/5}}{x^{2/5}} \cdot \dfrac{x^{3/5}}{x^{3/5}} = \dfrac{\sqrt[5]{4x^3}}{x}$

1.3 Algebra of Polynomials

A ***polynomial*** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \cdots a_1$ and a_0 are real numbers and n is a nonnegative integer. The ***degree*** of the polynomial is n if $a_n \neq 0$. Each of the (nonzero) expressions $a_n x^n, a_{n-1} x^{n-1}, \cdots a_1 x$ and a_0 is a ***term*** of the polynomial.

Polynomials are classified by the number of terms they contain and by their degree. A polynomial with one term is called a *monomial*, a polynomial with two terms is called a *binomial*, and a polynomial with three terms is called a *trinomial*. The following table provides some examples.

Polynomial	Type	Terms	Degree
$3x^2 - x + 5$	Trinomial	$3x^2,\ -x,\ 5$	2
$7 - x^4$	Binomial	$7,\ -x^4$	4
$x^5 - 2x^3 + 6x^2 - 1$	Four Terms	$x^5,\ -2x^3,\ 6x^2,\ -1$	5
$9x^8$	Monomial	$9x^8$	8
-12	Monomial	-12	0

Polynomials can have more than one variable. The degree of such a polynomial is the largest of the sums of the exponents of the variables in each term of the polynomial. The following table provides some examples.

Polynomial	Type	Terms	Degree
$2x^3 - 5y^2$	Binomial	$2x^3, -5y^2$	3
$x^5y^2 + \sqrt{2}z^6$	Binomial	$x^5y^2, \sqrt{2}z^6$	7
$z^8 - 2xy^4z^5 + x^2y^7$	Trinomial	$z^8, -2xy^4z^5, x^2y^7$	10

Although the polynomial $z^8 - 2xy^4z^5 + x^2y^7$ above has degree 10, we can think of it as a polynomial in z of degree 8 if we view the variables x and y as constants. Similarly, we can think of it as a polynomial in x of degree 2 or a polynomial in y of degree 7.

Polynomials can be added, subtracted, multiplied and factored. A key property of real numbers called the *Distributive Property* is the foundation for all these operations on polynomials.

Distributive Property
If a, b and c are real numbers, then

$$a(b+c) = ab + ac \quad \textit{and} \quad (b+c)a = ba + ca = ab + ac$$

The addition and subtraction of polynomials involves "combining like terms" and this technique, in turn, is an application of the Distributive Property. For example, when we add $2x^2$ and $3x^2$, we use the distributive property to obtain

$$2x^2+3x^2=(2+3)x^2=5x^2$$

The multiplication of a polynomial by a constant or another monomial is also an application of the Distributive Property. For example, the expansions

$$6(2x^2-3x+8)=12x^2-18x+48 \qquad \textit{and} \qquad 2x^3(x^2-4)=2x^5-8x^3$$

are both applications of the Distributive Property. The following example further illustrates how the Distributive Property is applied to add and subtract polynomials.

Example 1. Perform the indicated operations and simplify.

(a) $3(x^2-x+5)-2(4x^2-5x-3)$

(b) $3x^2(x-4)+4(x^3+7x^2-5)$

(c) $x^2-\frac{3}{8}x^2+\frac{5}{6}x^2$

Solution. **(a)** We first apply the Distributive Property to multiply each term in the first polynomial by 3 and each term in the second polynomial by -2 and then collect like terms to obtain

$$\begin{aligned} &3(x^2-x+5)-2(4x^2-5x-3) && \text{Given polynomial}\\ &\quad=3x^2-3x+15-8x^2+10x+6 && \text{Distributive Property}\\ &\quad=3x^2-8x^2-3x+10x+15+6 && \text{Group like terms}\\ &\quad=(3-8)x^2+(-3+10)x+(15+6) && \text{Combine like terms}\\ &\quad=-5x^2+7x+21 && \text{Simplify} \end{aligned}$$

We included the next to last step only to emphasize how the Distributive Property is used to combine like terms.

(b) We first use the Distributive Property to multiply each term in the first polynomial by $3x^2$ and each term in the second polynomial by 4 and collect like terms to obtain

$$\begin{aligned} &3x^2(x-4)+4(x^3+7x^2-5) && \text{Given polynomial}\\ &\quad=3x^3-12x^2+4x^3+28x^2-20 && \text{Distributive Property}\\ &\quad=7x^3+16x^2-20 && \text{Combine like terms} \end{aligned}$$

(c) All terms in this polynomial $x^2-\frac{3}{8}x^2+\frac{5}{6}x^2$ are like terms since they are all multiples of x^2. Two of the three coefficients are fractions and appear intimidating.

However, if we realize that the coefficient of the first term is 1 and that we can use the Distributive Property, then we understand that this problem is an exercise in adding fractions. We obtain

$x^2 - \frac{3}{8}x^2 + \frac{5}{6}x^2$	Given polynomial
$= \left(1 - \frac{3}{8} + \frac{5}{6}\right)x^2$	Distributive Property
$= \left(\frac{24}{24} - \frac{9}{24} + \frac{20}{24}\right)x^2$	Write each fraction with LCD 24
$= \frac{35}{24}x^2$	Add fractions

We saw in Example 1 how the Distributive Property was applied to find the product of a monomial and a polynomial. The Distributive Property is in fact used when multiplying *any* two polynomials. The most commonly encountered is the product of two binomials. For example, suppose we are asked to find $(2x-3)(4x+5)$. We think of $4x+5$ as a monomial and apply the Distributive Property and properties of exponents to obtain

$(2x-3)(4x+5) = 2x(4x+5) - 3(4x+5)$	Distributive Property
$= \underbrace{2x \cdot 4x}_{F} + \underbrace{2x \cdot 5}_{O} - \underbrace{3 \cdot 4x}_{I} - \underbrace{3 \cdot 5}_{L}$	Distributive Property
$= 8x^2 + 10x - 12x - 15$	Multiply factors
$= 8x^2 - 2x - 15$	Combine like terms

This gives the well-known ***FOIL*** method. We find the product of the ***F***irst terms in each binomial, add the product of the ***O***uter terms, add the product of the ***I***nner terms, and finally add the product of the ***L***ast terms. We complete the expansion by finding the indicated products and combining like terms.

You are encouraged to become proficient with the ***FOIL*** method to the point where you can carry out the calculations mentally (in most cases) without writing down any intermediate steps. This ability helps us factor polynomials efficiently as we will see in Section 1.4. But it is also important to understand that the ***FOIL*** method is based on the Distributive Property and *only* applies when multiplying two binomials.

The Distributive Property can be applied to find the product of any two algebraic expressions, not just two binomials. The examples that follow show how this is done.

Example 2. Perform the indicated operations and simplify.

(a) $(2x+3y)^2$ **(b)** $5(2a^2-3b^2)(4a^2+b^2)$

(c) $(x+2y)(x^2-3xy+y^2)$ **(d)** $\left(x^{3/2}+y^{1/2}\right)\left(x^{3/2}-y^{1/2}\right)$

(e) $\left(\sqrt{x}+\dfrac{2}{y}\right)\left(\sqrt{x}+\dfrac{3}{y}\right)$

Solution.* (a)** If we think about what $(2x+3y)^2$ means, then this calculation is a simple application of the ***FOIL method. We have

$(2x+3y)^2 = (2x+3y)(2x+3y)$	Definition of $(2x+3y)^2$
$= 2x\cdot 2x + 2x\cdot 3y + 3y\cdot 2x + 3y\cdot 3y$	***FOIL***
$= 4x^2 + 6xy + 6xy + 9y^2$	Multiply factors
$= 4x^2 + 12xy + 9y^2$	Combine like terms

Note: The incorrect squaring of a binomial is a very common error that you should try to avoid. Students often *incorrectly* write $(2x+3y)^2 = (2x)^2 + (3y)^2 = 4x^2 + 9y^2$. The error occurs by confusing *terms* with *factors*. Students apply the property of exponents for *factors* $(2x\cdot 3y)^2 = (2x)^2(3y)^2 = 4x^2\cdot 9y^2$ not realizing that they are dealing with *terms*.

(b) There are two strategies that could be used to expand the product $5(2a^2-3b^2)(4a^2+b^2)$. We can first multiply the two binomials, and then multiply the result by 5, or we could multiply one of the binomials by 5, and then multiply the resulting binomials. We choose the first strategy to obtain

$5(2a^2-3b^2)(4a^2+b^2) = 5\left(8a^4+2a^2b^2-12a^2b^2-3b^4\right)$	***FOIL***
$= 5\left(8a^4-10a^2b^2-3b^4\right)$	Combine like terms
$= 40a^4-50a^2b^2-15b^4$	Distributive Property

(c) We cannot use the ***FOIL*** method because we have a binomial multiplied by a trinomial. Instead, we apply the Distributive Property, properties of exponents and combine like terms to obtain

$(x+2y)(x^2-3xy+y^2) = x(x^2-3xy+y^2)+2y(x^2-3xy+y^2)$	Distributive Property
$= (x^3-3x^2y+xy^2)+(2x^2y-6xy^2+2y^3)$	Distributive Property
$= x^3-x^2y-5xy^2+2y^3$	Collect like terms

(d) The algebraic expressions involved in this example are not polynomials, but the Distributive Property and the *FOIL* method can still be used.

$\left(x^{3/2}+y^{1/2}\right)\left(x^{3/2}-y^{1/2}\right) = x^{3/2}x^{3/2}+x^{3/2}y^{1/2}-x^{3/2}y^{1/2}-y^{1/2}y^{1/2}$	*FOIL*
$= x^{3/2+3/2}-y^{1/2+1/2}$	Property of Exponents
$= x^3-y$	Simplify

(e) The algebraic expressions involved in this example are not polynomials, but the Distributive Property and the *FOIL* method can still be used.

$$\left(\sqrt{x}+\frac{2}{y}\right)\left(\sqrt{x}+\frac{3}{y}\right)=\sqrt{x}\cdot\sqrt{x}+\sqrt{x}\cdot\frac{3}{y}+\sqrt{x}\cdot\frac{2}{y}+\frac{2}{y}\cdot\frac{3}{y} \qquad \textit{FOIL}$$

$$=x+\frac{3\sqrt{x}}{y}+\frac{2\sqrt{x}}{y}+\frac{6}{y^2} \qquad \text{Multiply factors}$$

$$=x+\frac{5\sqrt{x}}{y}+\frac{6}{y^2} \qquad \text{Combine like terms}$$

Rationalizing Numerator or Denominator of an Expression

Definition 1. The *conjugate* of an algebraic or numeric expression of the form $X+Y$ is $X-Y$. It follows that the conjugate of $X-Y$ is $X+Y$.

Definition 2. *Rationalizing the denominator* of an algebraic or numeric fraction means writing it as an equivalent fraction without any radical signs in the denominator. *Rationalizing the numerator* of an algebraic or numeric fraction means writing it as an equivalent fraction without any radical signs in the numerator.

The conjugate plays an important role in rationalizing the numerator or denominator of an algebraic or numeric fraction as discussed in the following examples.

Example 3. Rationalize the denominator of the expression.

(a) $\dfrac{5}{3-\sqrt{2}}$ **(b)** $\dfrac{1}{\sqrt{x}+\sqrt{y}}$

Solution. (a) We multiply the numerator and denominator by the conjugate $3+\sqrt{2}$ of $3-\sqrt{2}$ to obtain

$$\frac{5}{3-\sqrt{2}}=\frac{5}{3-\sqrt{2}}\cdot\frac{3+\sqrt{2}}{3+\sqrt{2}} \qquad \text{Multiply the numerator and denominator by the conjugate of } 3-\sqrt{2}$$

$$=\frac{5(3+\sqrt{2})}{3^2-(\sqrt{2})^2} \qquad (3-\sqrt{2})(3+\sqrt{2})=3^2-(\sqrt{2})^2$$

$$=\frac{15+5\sqrt{2}}{7} \qquad \text{Simplify}$$

(b) We multiply the numerator and denominator by the conjugate $\sqrt{x}-\sqrt{y}$ of $\sqrt{x}+\sqrt{y}$ to obtain

$$\frac{1}{\sqrt{x}+\sqrt{y}} = \frac{1}{\sqrt{x}+\sqrt{y}} \cdot \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

Multiply the numerator and denominator by the conjugate of $\sqrt{x}+\sqrt{y}$

$$= \frac{\sqrt{x}-\sqrt{y}}{(\sqrt{x})^2-(\sqrt{y})^2}$$

$(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = (\sqrt{x})^2-(\sqrt{y})^2$

$$= \frac{\sqrt{x}-\sqrt{y}}{x-y}$$

Simplify

Example 4. Rationalize the numerator of the expression.

(a) $\dfrac{\sqrt{3}-\sqrt{5}}{2}$ **(b)** $\sqrt{x^2+1}+x$

Solution. **(a)** We multiply the numerator and denominator by the conjugate $\sqrt{3}+\sqrt{5}$ of $\sqrt{3}-\sqrt{5}$ to obtain

$$\frac{\sqrt{3}-\sqrt{5}}{2} = \frac{\sqrt{3}-\sqrt{5}}{2} \cdot \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}}$$

Multiply numerator and denominator by the conjugate of $\sqrt{3}-\sqrt{5}$

$$= \frac{(\sqrt{3})^2-(\sqrt{5})^2}{2(\sqrt{3}+\sqrt{5})}$$

$(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5}) = (\sqrt{3})^2-(\sqrt{5})^2$

$$= -\frac{1}{\sqrt{3}+\sqrt{5}}$$

Simplify

(b) We multiply the numerator and denominator by the conjugate $\sqrt{x^2+1}-x$ of $\sqrt{x^2+1}+x$ to obtain

$$\sqrt{x^2+1}+x = \frac{\sqrt{x^2+1}+x}{1} \cdot \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}$$

Multiply numerator and denominator by the conjugate of $\sqrt{x^2+1}+x$

$$= \frac{(\sqrt{x^2+1})^2-x^2}{\sqrt{x^2+1}-x}$$

$(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x) = (\sqrt{x^2+1})^2-x^2$

$$= \frac{1}{\sqrt{x^2+1}-x}$$

Simplify

Evaluating a Polynomial

Example 5. Find the value of the polynomial expression x^2-4x+5 when $x=1-\sqrt{3}$. Write your answer in the form $A+B\sqrt{3}$ where A and B are integers.

Solution. We substitute $1-\sqrt{3}$ for x in x^2-4x+5 to obtain

x^2-4x+5	Given polynomial expression
$(1-\sqrt{3})^2-4(1-\sqrt{3})+5$	Substitute $x=1-\sqrt{3}$
$(1-\sqrt{3})(1-\sqrt{3})-4(1-\sqrt{3})+5$	$(1-\sqrt{3})^2=(1-\sqrt{3})(1-\sqrt{3})$
$\left(1-2\sqrt{3}+(\sqrt{3})^2\right)-4(1-\sqrt{3})+5$	*FOIL*
$1-2\sqrt{3}+3-4+4\sqrt{3}+5$	Expand
$5+2\sqrt{3}$	Combine like terms

Thus the value of x^2-4x+5 when $x=1-\sqrt{3}$ is $5+2\sqrt{3}$.

1.4 Factoring Polynomials

When we multiplied polynomials, we took expressions in parentheses in the form $a(b+c)$ and expanded them to remove the parentheses using the Distributive Property to obtain $ab+ac$. When we factor a polynomial in the form $ab+ac$, we reverse this process. We factor out the common factor a from both terms and use the Distributive Property to obtain $a(b+c)$.

Factoring Out the Greatest Common Factor (GCF)

Suppose we are asked to factor the polynomial $24x^3y^7w^2+90x^5y^4z$. We begin by observing that x^3y^4 is greatest common factor (*GCF)* of $x^3y^7w^2$ and x^5y^4z. If the *GCF* of 24 and 90 is not immediately apparent, we can break these numbers into their prime factorizations $24=2^3\cdot 3$ and $90=2\cdot 3^2\cdot 5$. We then see that the GCF of 24 and 90 is $2\cdot 3=6$. We factor out $6x^3y^4$ from both terms and apply the Distributive Property to obtain the desired factorization.

$$\begin{aligned} 24x^3y^7w^2+90x^5y^4z &= 6x^3y^4\cdot 4y^3w^2+6x^3y^4\cdot 15x^2z && \text{Factor } 6x^3y^4 \text{ from each term} \\ &= 6x^3y^4(4y^3w^2+15x^2z) && \text{Factor out } 6x^3y^4 \end{aligned}$$

Example 1. Find the greatest common factor (*GCF*) of the terms in the expression. Write the expression by factoring out the *GCF* in each of its terms and then use the distributive law to write the expression in factored form.

(a) $8x^3-12x^2+36x$ **(b)** $15a^2(b-1)-18a^3(b-1)$

***Solution*. (a)** The *GCF* of the three terms $8x^3$, $12x^2$ and $36x$ is $4x$. We factor out $4x$ from both terms and apply the Distributive Property to obtain

$$\begin{aligned} 8x^3-12x^2+36x &= 4x\cdot 2x^2-4x\cdot 3x+4x\cdot 9 && \text{Factor } 4x \text{ from each term} \\ &= 4x(2x^2-3x+9) && \text{Factor out } 4x \end{aligned}$$

(b) The *GCF* of $42a^2(b-1)$ and $18a^3(b-1)$ is $6a^2(b-1)$. We factor out $6a^2(b-1)$ from both terms and apply the Distributive Property to obtain

$$\begin{aligned} &42a^2(b-1)-18a^3(b-1) \\ &\quad = 6a^2(b-1)\cdot 7-6a^2(b-1)\cdot 3a && \text{Factor } 6a^2(b-1) \text{ from each term} \\ &\quad = 6a^2(b-1)(7-3a) && \text{Factor out } 6a^2(b-1) \end{aligned}$$

Factoring Trinomials with Leading Coefficient 1

We use a trial and error method to factor trinomials of the form x^2+bx+c where b and c are integers.

Example 2. Factor: **(a)** x^2-6x+8 **(b)** x^2+9x+8 **(c)** x^2+4x+8

Solution. All three trinomials have a constant term of 8. Factorization of these trinomials will have the form

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

where a and b are integers with $ab=8$. There are four possible factorizations of 8: $8\cdot 1$, $(-8)\cdot(-1)$, $4\cdot 2$, $(-4)\cdot(-2)$. Thus there are four possible factorizations of each trinomial:

$$(x+8)(x+1),\ (x-8)(x-1),\ (x+4)(x+2),\ (x-4)(x-2)$$

(a) We factor x^2-6x+8 by strategically choosing one of these four possibilities. The only choice whose expansion gives a middle term of $-6x$ is $(x-4)(x-2)$. Therefore $x^2-6x+8=(x-4)(x-2)$.

(b) We factor x^2+9x+8 by choosing $(x+8)(x+1)$ since its expansion has a middle term of $9x$. Thus $x^2+9x+8=(x+8)(x+1)$.

(c) None of the four possible factorizations has an expansion with a middle term of $4x$. Thus x^2+4x+8 cannot be factored.

Example 3. Factor: **(a)** $x^2+4x-12$ **(b)** x^2-x-12 **(c)** $x^2+2x-12$

Solution. All three trinomials have a constant term of -12. and so the same trial and error process can be used to factor them. The only difference between this example and Example 1 is that there are more possible binomial factors. The following table shows the possible factors of -12, the corresponding possible binomial factors and the middle term resulting when these factors are multiplied.

Factors of -12	**Possible Binomial Factors**	**Middle Term**
$12,-1$	$(x+12)(x-1)$	$11x$
$-12,1$	$(x-12)(x+1)$	$-11x$
$6,-2$	$(x+6)(x-2)$	$4x$
$-6,2$	$(x-6)(x+2)$	$-4x$
$4,-3$	$(x+4)(x-3)$	x
$-4,3$	$(x-4)(x+3)$	$-x$

(a) We see from the table that $x^2+4x-12=(x+6)(x-2)$since $4x$is the middle term we seek.
(b) We see from the table that $x^2-x-12=(x-4)(x+3)$since $-x$ is the middle term we seek.
(c) The table lists the *possible* binomial factors of $x^2+2x-12$. Since none of these factors has a middle term $2x$, $x^2+2x-12$ cannot be factored.

Note. The trial and error process of factoring trinomials requires more work than the reverse process of multiplying two binomials. The examples above give a systematic way to implement the trial and error process. However, you should not in general have to write out an entire table to factor trinomials of this kind. Instead, much of the work in finding the possible binomial factors and the middle terms can be done mentally with practice.

Factoring Trinomials with Leading Coefficient Not 1

We use a trial and error method discussed in the next two examples.

Example 4. Factor: **(a)** $3x^2-13x-10$ **(b)** $3x^2+7x-10$ **(c)** $3x^2-5x-10$

Solution. All three trinomials have the same first term and last term and so the same trial and error process can be used to factor them. The following table shows the factors of 3, the coefficient of x^2, and -10, the constant term.

Factors of 3	**Factors of -10**	**Possible Binomial Factors**	**Middle Term**
$3,1$	$10,-1$	$(3x+10)(x-1)$ $(3x-1)(x+10)$	$7x$ $29x$
$3,1$	$-10,1$	$(3x-10)(x+1)$ $(3x+1)(x-10)$	$-7x$ $-29x$
$3,1$	$5,-2$	$(3x+5)(x-2)$ $(3x-2)(x+5)$	$-x$ $13x$
$3,1$	$-5,2$	$(3x-5)(x+2)$ $(3x+2)(x-5)$	x $-13x$

Notice that for each pair of factors of -10 there are two possible binomial factors. For example, the factors $5,-2$ correspond to the two binomial factors $(3x+5)(x-2)$ and $(3x-2)(x+5)$. The large number of possible factors makes it all the more important that you can calculate the middle term quickly. We suggest that you go through eight binomial factors in the table, calculate the middle term mentally, and check with the answer in the table.

(a) We see from the table that $3x^2-13x-10=(3x+2)(x-5)$ since $-13x$ is the middle term we seek.

(b) We see from the table that $3x^2+7x-10=(3x+10)(x-1)$ since $7x$ is the middle term we seek.

(c) The table lists the *possible* binomial factors of $3x^2-5x-10$. Since none of these factors has a middle term $-5x$, $3x^2-5x-10$ cannot be factored.

Example 5. Factor: **(a)** $6x^2-13x+5$ **(b)** $6x^2-11x+5$ **(c)** $6x^2+9x+5$

Solution. All three trinomials have the same first term and last term and so the same trial and error process can be used to factor them. The following table shows the factors of 6, the coefficient of x^2, and 5, the constant term.

Factors of 6	Factors of 5	Possible Binomial Factors	Middle Term
$6,1$	$5,1$	$(6x+5)(x+1)$ $(6x+1)(x+5)$	$11x$ $31x$
$6,1$	$-5,-1$	$(6x-5)(x-1)$ $(6x-1)(x-5)$	$-11x$ $-31x$
$3,2$	$5,1$	$(3x+5)(2x+1)$ $(3x+1)(2x+5)$	$13x$ $17x$
$3,2$	$-5,-1$	$(3x-5)(2x-1)$ $(3x-1)(2x-5)$	$-13x$ $-17x$

(a) We see from the table that $6x^2-13x+5=(3x-5)(2x-1)$ since $-13x$ is the middle term we seek.

(b) We see from the table that $6x^2-11x+5=(6x-5)(x-1)$ since $-11x$ is the middle term we seek.

(c) The table lists the *possible* binomial factors of $6x^2+9x+5$. Since none of these factors has a middle term $9x$, $6x^2+9x+5$ cannot be factored.

Factoring Difference of Squares

A formula can be used to factor a few special polynomials. The most important such formula is the difference of squares.

Difference of Squares Formula

$$A^2-B^2=(A-B)(A+B)$$

Example 6. Factor: **(a)** $25x^2-9$ **(b)** x^2-4y^2 **(c)** a^4-16b^4

Solution. **(a)** We recognize $25x^2-9=(5x)^2-(3)^2$ as the difference of squares with $A=5x$ and $B=3$. We use the difference of squares formula to obtain

$$\begin{aligned} 25x^2-9&=(5x)^2-(3)^2 && \text{Write each term as a square} \\ &=(5x-3)(5x+3) && \text{Difference of Squares Formula} \end{aligned}$$

(b) We recognize $x^2 - 4y^2 = (x)^2 - (2y)^2$ as the difference of squares with $A = x$ and $B = 2y$. We use the difference of squares formula to obtain

$x^2 - 4y^2 = (x)^2 - (2y)^2$	Write each term as a square
$= (x-2y)(x+2y)$	Difference of Squares Formula

(c) We recognize $a^4 - 16b^4 = (a^2)^2 - (4b^2)^2$ as the difference of squares with $A = a^2$ and $B = 4b$. We use the difference of squares formula to obtain

$a^4 - 16b^4 = (a^2)^2 - (4b^2)^2$	Write each term as a square
$= (a^2 - 4b^2)(a^2 + 4b^2)$	Difference of Squares Formula

We then observe that $a^2 - 4b^2 = (a-2b)(a+2b)$ is also the difference of squares and we write the complete factorization as

$$a^4 - 4b^4 = (a-2b)(a+2b)(a^2+4b^2)$$

Factoring a Polynomial Completely

A complete factorization of a polynomial often requires more than one step. When we factor a polynomial, *we first factor out the greatest common factor*, then inspect the result to see if further factoring is possible.

Example 7. Factor the given polynomial completely.

(a) $8x^3y^2 - 2xy^4$ **(b)** $3x^2 - 6xy - 24y^2$

(c) $4a^3 + 10a^2 - 6a$ **(d)** $x^2(y^2-1) - 25(y^2-1)$

Solution. **(a)** We first factor out the *GCF* $2xy^2$ of the two terms in $8x^3y^2 - 2xy^4$ and then recognize that the resulting factor $4x^2 - y^2$ is the difference of squares to obtain

$8x^3y^2 - 2xy^4 = 2xy^2(4x^2 - y^2)$	Factor out $2xy^2$
$= 2xy^2(2x-y)(2x+y)$	Factor: $4x^2 - y^2$

(b) We first factor out the *GCF* 3 of the three terms in $3x^2 - 6xy - 24y^2$ and then recognize that the resulting factor $x^2 - 2xy - 8y^2$ is a trinomial that we can factor. We obtain

$3x^2 - 6xy - 24y^2 = 3(x^2 - 2xy - 8y^2)$	Factor out 3
$= 3(x-4y)(x+2y)$	Factor: $x^2 - 2xy - 8y^2$

(c) We first factor out the *GCF* $2a$ of the three terms in $4a^3+10a^2-6a$ and then recognize that the resulting factor $2a^2+5a-3$ is a trinomial that we can factor. We obtain

$$\begin{aligned} 4a^3+10a^2-6a &= 2a(2a^2+5a-3) && \text{Factor out } 2a \\ &= 2a(2a-1)(a+3) && \text{Factor: } 2a^2+5a-3 \end{aligned}$$

(d) We first realize that the *binomial* y^2-1 is a common factor (and indeed the greatest common factor) of the two terms in $x^2(y^2-1)-25(y^2-1)$. We factor it out to obtain

$$x^2(y^2-1)-25(y^2-1)=(x^2-25)(y^2-1) \qquad \text{Factor out } y^2-1$$

We then recognize that both x^2-25 and y^2-1 are the difference of squares. We complete the factorization to obtain

$$\begin{aligned} x^2(y^2-1)-25(y^2-1) &= (x^2-25)(y^2-1) \\ &= (x-5)(x+5)(y-1)(y+1) && \text{Factor } x^2-25,\ y^2-1 \end{aligned}$$

1.5 More on Factoring

We discuss methods for factoring some additional kinds of polynomials in this section. We discussed factoring the difference of two squares in the previous section. In this section, we discuss factoring the sum and difference of two cubes using the special factoring formulas shown below.

Special Factoring Formulas

Formula	*Name*
1. $A^2 - B^2 = (A-B)(A+B)$	Difference of Squares
2. $A^3 - B^3 = (A-B)(A^2+AB+B^2)$	Difference of Cubes
3. $A^3 + B^3 = (A+B)(A^2-AB+B^2)$	Sum of Cubes

We note that the *sum* of two squares of the form A^2+B^2 *cannot* be factored.

Factoring the Difference and Sum of Cubes

Example 1. Factor: **(a)** $8x^3+27y^3$ **(b)** $3x^3-24$ **(c)** $54x^6-16y^6$

Solution. **(a)** We recognize $8x^3+27y^3=(2x)^3+(3y)^3$ as the sum of cubes with $A=2x$ and $B=3y$. We use the sum of cubes formula to obtain

$$\begin{aligned} 8x^3+27y^3 &= (2x)^3+(3y)^3 && \text{Write each term as a cube} \\ &= (2x+3y)\left[(2x)^2-(2x)(3y)+(3y)^2\right] && \text{Sum of Cubes Formula} \\ &= (2x+3y)(4x^2-6xy+9y^2) && \text{Simplify} \end{aligned}$$

(b) We begin by factoring out the *GCF* 3 to obtain $3x^3-24=3(x^3-8)$. We then recognize that x^3-8 is the difference of cubes with $A=x$ and $B=2$ to obtain

$$\begin{aligned} 3x^3-24 &= 3(x^3-8) && \text{Factor out } GCF\ 3 \\ &= 3\left[(x-2)(x^2+2x+2^2)\right] && \text{Difference of Cubes Formula} \\ &= 3(x-2)(x^2+2x+4) && \text{Simplify} \end{aligned}$$

(c) We begin by factoring out the *GCF* 2 to obtain $54x^6-16y^6=2(27x^6-8y^6)$. We then recognize that $27x^6-8y^6$ is the difference of cubes with $A=3x^2$ and $B=2y^2$ to obtain

$$\begin{aligned} 54x^6-16y^6 &= 2(27x^6-8y^6) && \text{Factor out } GCF\ 2 \\ &= 2\left[(3x^2-2y^2)((3x^2)^2+(3x^2)(2y^2)+(2y^2)^2\right] && \text{Difference of Cubes Formula} \\ &= 2(3x^2-2y^2)(9x^4+6x^2y^2+4y^4) && \text{Simplify} \end{aligned}$$

Factoring by Grouping

A polynomial with four terms can in some special cases be factored by a method called grouping. This method works when the factored form of the first two terms has a factor in common with the factored form of the second two terms. We illustrate this method in Example 2.

Example 2. Factor the given polynomial by grouping.

(a) $x^3 - 2x^2 - 3x + 6$ **(b)** $3x^5 + 12x^4 + 9x^3 + 36x^2$

Solution. **(a)** We notice that the first two terms of $x^3 - 2x^2 - 3x + 6$ can be factored as $x^3 - 2x^2 = x^2(x-2)$, the last two as $-3x + 6 = -3(x-2)$ and both have a common factor of $x - 2$. We then have

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) + (-3x + 6) && \text{Group terms} \\ &= x^2(x-2) - 3(x-2) && \text{Factor grouped terms} \\ &= (x-2)(x^2-3) && \text{Factor out } x-2 \end{aligned}$$

(b) We begin by factoring out the *GCF* $3x^2$ of all four terms. We then group the remaining four terms to obtain

$$\begin{aligned} 3x^5 + 12x^4 + 9x^3 + 36x^2 &= 3x^2\left[(x^3 + 4x^2) + (3x + 12)\right] && \text{Factor out } GCF \\ &= 3x^2\left[x^2(x+4) + 3(x+4)\right] && \text{Factor } x^3 + 4x^2,\ 3x + 12 \\ &= 3x^2\left[(x+4)(x^2+3)\right] && \text{Factor out } x+4 \\ &= 3x^2(x+4)(x^2+3) && \text{Simplify} \end{aligned}$$

Seeing Structure in Polynomial Expressions

Seeing the structure of a polynomial can help us view it as a simpler polynomial that we can factor. For example, we can see that $x^4 - y^4 = (x^2)^2 - (y^2)^2$ is the difference of squares and can factor it using the difference of squares formula. Likewise, we can see that $x^6 + 8y^6 = (x^2)^3 + (2y^2)^3$ is the sum of cubes and factor it using the sum of cubes formula. Example 3 provides some additional examples.

Example 3. Factor: **(a)** $x^6 - 4x^3 - 12$ **(b)** $3(x+1)^2 + 2(x+1) - 21$

Solution. **(a)** We observe that $x^6 - 4x^3 - 12$ can be viewed as a quadratic polynomial with x^3 as the variable. We then have

$$\begin{aligned} x^6 - 4x^3 - 12 &= (x^3)^2 - 4(x^3) - 12 && \text{Write as quadratic polynomial} \\ &= (x^3 - 6)(x^3 + 2) && \text{Factor} \end{aligned}$$

(b) We observe that $3(x+1)^2+2(x+1)-21$ can be viewed as a quadratic polynomial with $x+1$ as the variable. We then have

$$\begin{aligned} 3(x+1)^2+2(x+1)-21 &= [3(x+1)-7][(x+1)+3] && \text{Factor quadratic polynomial} \\ &= (3x-4)(x+4) && \text{Simplify} \end{aligned}$$

Factoring Strategies

The following factoring strategies will be helpful in factoring polynomials.

Step 1. Factor out the greatest common factor (*GCF*) of the terms in the polynomial.
Step 2. Examine the polynomial remaining after the *GCF* has been factored out. (This polynomial will be the original polynomial if the *GCF* is 1.) Consider the number of terms in this polynomial.

- If this polynomial has exactly two terms, determine whether it is the difference of squares, the difference of cubes, or the sum of cubes, and, if so, use the appropriate formula to factor it.
- If the polynomial has exactly three terms, use the trial and error method for trinomials to factor it.
- If the polynomial has exactly four terms, use the method of grouping terms to factor it.

1.6 Rational Expressions

Definition 1. A **rational expression** is the quotient $\dfrac{P}{Q}$ of two polynomials P and Q in one or more variables, where $Q \neq 0$.

Some examples of rational expressions are:

$$\frac{x}{x^2+2x-4} \qquad \frac{ab^2}{c^4} \qquad \frac{x^3-y^3}{x^2+y^2}$$

Rational expressions in algebra are closely related to *rational* numbers in arithmetic as their names and definitions suggest. Recall that a rational number is the quotient $\dfrac{p}{q}$ of two integers p and q, where $q \neq 0$. The procedure used to simplify a rational expression by dividing out the greatest common factor of the numerator and denominator is the same as the procedure used to reduce a rational number (a fraction) to lowest terms. The procedures used to multiply, divide, add and subtract rational expressions are the same as the corresponding procedures used to multiply, divide, add and subtract rational numbers.

Simplifying Rational Expressions

Example 1. Find the greatest common factor (*GCF*) of the numerator and the denominator of each rational expression. Write the expression by factoring out the *GCF* in both the numerator and the denominator. Then divide out the *GCF* to write the rational expression in simplified form.

(a) $\dfrac{18a^2b^5c^3}{12a^4b}$ **(b)** $\dfrac{9x^2(2x-1)^3}{24x^5(2x-1)}$

Solution. **(a)** The *GCF* of the numerator and denominator of $\dfrac{18a^2b^5c^3}{12a^4b}$ is $6a^2b$. We factor out the *GCF* of the numerator and denominator and divide it out to obtain

$$\begin{aligned} \frac{18a^2b^5c^3}{12a^4b} &= \frac{6a^2b\cdot 3b^4c^3}{6a^2b\cdot 2a^2} && \text{Factor } 6a^2b \text{ from numerator and denominator} \\ &= \frac{3b^4c^3}{2a^2} && \text{Divide out } GCF\ 6a^2b \end{aligned}$$

(b) The *GCF* of the numerator and denominator of $\dfrac{9x^2(2x-1)^3}{24x^5(2x-1)}$ is $3x^2(2x-1)$. We factor out the *GCF* of the numerator and denominator and divide it out to obtain

$$\frac{9x^2(2x-1)^3}{24x^5(2x-1)} = \frac{3x^2(2x-1)\cdot 3(2x-1)^2}{3x^2(2x-1)\cdot 8x^3}$$ Factor $3x^2(2x-1)$ from numerator and denominator

$$= \frac{3(2x-1)^2}{8x^3}$$ Divide out *GCF* $3x^2(2x-1)$

Example 2. Simplify the given rational expression.

(a) $\dfrac{3a^2b}{3a^2b+6ab}$ **(b)** $\dfrac{x^2-x-6}{3x^2+5x-2}$ **(c)** $\dfrac{a^4b-ab^4}{a^3-ab^2}$ **(d)** $\dfrac{y^2-x^2}{x^2-xy}$

Solution. **(a)** We factor the denominator of $\dfrac{3a^2b}{3a^2b+6ab}$ and divide out the *GCF* $3ab$ of numerator and denominator to obtain

$$\frac{3a^2b}{3a^2b+6ab} = \frac{3a^2b}{3ab(a+2)}$$ Factor $3ab$ from numerator and denominator

$$= \frac{a}{a+2}$$ Divide out *GCF* $3ab$

Note: *Remember that you can only divide out, or cancel, an expression if it is **a factor of both the numerator and denominator** of a rational expression.* A common mistake in this case is to assume that $3a^2b$ is a *factor* of the denominator instead of a term and "cancel" it out to obtain the incorrect simplification $\dfrac{1}{6ab}$.

(b) We factor the numerator and denominator polynomials of $\dfrac{x^2-x-6}{3x^2+5x-2}$ and divide out the common factor $x+2$ to obtain

$$\frac{x^2-x-6}{3x^2+5x-2} = \frac{(x-3)(x+2)}{(3x-1)(x+2)}$$ Factor numerator and denominator polynomials

$$= \frac{x-3}{3x-1}$$ Divide out $x+2$

(c) We factor the numerator and denominator polynomials of $\dfrac{a^4b-ab^4}{a^3-ab^2}$ and simplify to obtain

$$\frac{a^4b-ab^4}{a^3-ab^2} = \frac{ab(a^3-b^3)}{a(a^2-b^2)}$$ Factor GCF from numerator and denominator polynomials

$$= \frac{ab(a-b)(a^2+ab+b^2)}{a(a-b)(a+b)}$$ Factor a^3-b^3, a^2-b^2

$$= \frac{b(a^2+ab+b^2)}{a+b}$$ Simplify

(d) We factor the numerator and denominator polynomials of $\frac{y^2-x^2}{x^2-xy}$ and simplify to obtain

$$\frac{y^2-x^2}{x^2-xy}=\frac{(y-x)(y+x)}{x(x-y)}$$ Factor numerator and denominator polynomials

$$=\frac{-(x-y)(y+x)}{x(x-y)}$$ Write $y-x=-(x-y)$

$$=-\frac{y+x}{x}$$ Simplify

Multiplying and Dividing Rational Expressions

The procedures used to multiply and divide rational expressions are the same as those used to multiply and divide fractions. For example, if A, B, C and D are polynomials with $B\neq 0$ and $C\neq 0$, then

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{A\cdot C}{B\cdot D} \qquad \textit{and} \qquad \frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}=\frac{A\cdot D}{B\cdot C}$$

Once these operations are performed, the only remaining task is to simply the resulting rational expression. A few examples should illustrate this process.

Example 3. Perform the multiplication or division of the given rational expressions and simplify.

(a) $\frac{x^2+2x-8}{x^2-9}\cdot\frac{x^2+3x}{x^2-5x+6}$ **(b)** $\frac{x^5}{5x-10}\div\frac{x^2}{2x-4}$

(c) $\frac{3x^2-7x-6}{x^2-6x+9}\cdot\frac{5x^3-15x^2-2x+6}{3x^2+5x+2}$

Solution. **(a)** We factor the numerators and denominators of both rational expressions in the product and simplify the resulting rational expression to obtain

$$\frac{x^2+2x-8}{x^2-9}\cdot\frac{x^2+3x}{x^2-5x+6}$$ Given product of rational expressions

$$=\frac{(x-2)(x+4)}{(x-3)(x+3)}\cdot\frac{x(x+3)}{(x-2)(x-3)}$$ Factor polynomials

$$=\frac{(x-2)(x+4)x(x+3)}{(x-3)(x+3)(x-2)(x-3)}$$ Multiply

$$=\frac{x(x+4)}{(x-3)^2}$$ Divide out (cancel) $x-2$, $x+3$

(b) We invert the second expression $\dfrac{x^2}{2x-4}$ in the quotient $\dfrac{x^5}{5x-10} \div \dfrac{x^2}{2x-4}$, multiply it by the first expression $\dfrac{x^5}{5x-10}$ and simplify to obtain

$\dfrac{x^5}{5x-10} \div \dfrac{x^2}{2x-4}$	Given quotient of rational expressions
$= \dfrac{x^5}{5x-10} \cdot \dfrac{2x-4}{x^2}$	Invert $\dfrac{x^2}{2x-4}$ and multiply
$= \dfrac{x^5 \cdot 2(x-2)}{5(x-2) \cdot x^2}$	Multiply and factor $2x-4$, $5x-10$
$= \dfrac{2x^3}{5}$	Divide out (cancel) $x-2$, x^2

(c) We factor the numerators and denominators of both rational expressions and simplify to obtain

$\dfrac{3x^2-7x-6}{x^2-6x+9} \cdot \dfrac{5x^3-15x^2-2x+6}{3x^2+5x+2}$	Given product of rational expressions
$= \dfrac{(3x+2)(x-3)}{(x-3)^2} \cdot \dfrac{5x^2(x-3)-2(x-3)}{(3x+2)(x+1)}$	Factor polynomials
$= \dfrac{(3x+2)(x-3) \cdot (5x^2-2)(x-3)}{(x-3)^2 \cdot (3x+2)(x+1)}$	Complete the factorization and multiply
$= \dfrac{5x^2-2}{x+1}$	Divide out common factors

Adding and Subtracting Rational Expressions

The procedures used to add and subtract rational expressions are the same as those used to add and subtract fractions. Two rational expressions with the same denominator are relatively easy to add and subtract. For example, if A, B and C are polynomials with $C \neq 0$, then

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C} \qquad \textit{and} \qquad \frac{A}{C} - \frac{B}{C} = \frac{A-B}{C}$$

The procedure is more difficult to carry out if the denominators are different. For example, suppose A, B, C and D are polynomials with $C \neq 0$ and $D \neq 0$. To add the rational expressions $\dfrac{A}{C}$ and $\dfrac{B}{D}$, we must first find a common denominator, CD in this case. We then find equivalent expressions for $\dfrac{A}{C}$ and $\dfrac{B}{D}$ with this denominator and add to obtain

$$\frac{A}{C}+\frac{B}{D}=\frac{AD}{CD}+\frac{BC}{DC}=\frac{AD+BC}{CD}$$

In practice, it is important to find the *least common denominator*, or *LCD*, because otherwise the algebra becomes messy and it is difficult to reduce the rational expression that is obtained. This is the same approach used to add fractions with different denominators.

Example 4. Suppose the polynomials given are denominators of rational expressions. Find their least common denominator (*LCD*).

(a) $6x^3y^4, 8x^5y$ **(b)** $x^2-9, x^2-2x-15$ **(c)** $5(a+1)^2, 16(a+1)^3, 10(a+1)$

Solution. **(a)** To find the *LCD* of the denominators $6x^3y^4$ and $8x^5y$, we factor $6=2\cdot 3$ and $8=2^3$ into products of powers of prime numbers. We then examine these denominators in factored form:

$$2\cdot 3x^3y^4, \quad 2^3x^5y$$

We view the variables x and y as prime numbers and we take the largest power of each prime in the two expressions to form the *LCD*. The *LCD* of $6x^3y^4$ and $8x^5y$ is therefore

$$2^3\cdot 3x^5y^4=24x^5y^4$$

(b) To find the *LCD* of the denominators x^2-9 and $x^2-2x-15$, we factor them to obtain $x^2-9=(x-3)(x+3)$ and $x^2-2x-15=(x-5)(x+3)$. The *LCD* is therefore the product

$$(x-3)(x+3)(x-5)$$

Note. The product $(x-3)(x+3)^2(x-5)$ is *a common denominator* but not *the least common denominator*.

(c) To find the *LCD* of the denominators $5(a+1)^2$, $16(a+1)^3$ and $10(a+1)$, we factor $16=2^4$ and $10=2\cdot 5$ into products of powers of primes and examine these denominators in factored form:

$$5(a+1)^2, \quad 2^4(a+1)^3, \quad 2\cdot 5(a+1)$$

We take the largest power of each factor and multiply them together to obtain their *LCD*

$$2^4\cdot 5(a+1)^3=80(a+1)^3$$

Example 5. Find the *LCD* of the given pair of rational expressions. Express each rational expression in the pair as an equivalent rational expression with the *LCD* as its denominator.

(a) $\frac{3}{4a^2b}, \frac{5}{6ab^3}$ **(b)** $\frac{x}{x^2-x-2}, \frac{x+4}{x^2+3x-10}$

Solution. **(a)** The *LCD* of the denominators $4a^2b$ and $6ab^3$ is $12a^2b^3$. Thus $12a^2b^3$ is a multiple of both $4a^2b$ and $6ab^3$, and we can write

$$12a^2b^3 = 4a^2b \cdot 3b^2 \qquad 12a^2b^3 = 6ab^3 \cdot 2a$$

We can then write $\frac{3}{4a^2b}$ and $\frac{5}{6ab^3}$ as equivalent rational expressions with the same denominator $12a^2b^3$:

$$\frac{3}{4a^2b} = \frac{3 \cdot 3b^2}{4a^2b \cdot 3b^2} = \frac{9b^2}{12a^2b^3} \qquad \frac{5}{6ab^3} = \frac{5 \cdot 2a}{6ab^3 \cdot 2a} = \frac{10a}{12a^2b^3}$$

(b) We must factor the denominator polynomials $x^2 - x - 2 = (x-2)(x+1)$ and $x^2 + 3x - 10 = (x-2)(x+5)$ to find their *LCD*. Their *LCD* is therefore $(x-2)(x+1)(x+5)$. We can then write $\frac{x}{x^2-x-2}$ and $\frac{x+4}{x^2+3x-10}$ as equivalent rational expressions with the same denominator $(x-2)(x+1)(x+5)$.

$$\frac{x}{x^2-x-2} = \frac{x}{(x-2)(x+1)} = \frac{x(x+5)}{(x-2)(x+1)(x+5)}$$

$$\frac{x+4}{x^2+3x-10} = \frac{x+4}{(x-2)(x+5)} = \frac{(x+4)(x+1)}{(x-2)(x+5)(x+1)}$$

Example 6. Perform the addition or subtraction and simplify. Identify the *LCD* in each case.

(a) $\frac{5x}{6} + \frac{3x}{8}$ **(b)** $\frac{1}{2a} + \frac{1}{3a^2}$ **(c)** $\frac{2}{x-2} - \frac{3}{x+1}$

Solution. **(a)** The *LCD* of the denominators 6 and 8 is 24. Thus 24 is a multiple of both 6 and 8, and we can write $24 = 6 \cdot 4$ and $24 = 8 \cdot 3$. We write both $\frac{5x}{6}$ and $\frac{3x}{8}$ as equivalent expressions with the same denominator of 24 and add to obtain

$$\frac{5x}{6}+\frac{3x}{8}=\frac{5x\cdot 4}{6\cdot 4}+\frac{3x\cdot 3}{8\cdot 3}$$ Write each term as an equivalent expression with LCD 24

$$=\frac{20x}{24}+\frac{9x}{24}$$ Simplify

$$=\frac{29x}{24}$$ Add

(b) The *LCD* of the denominators $2a$ and $3a^2$ is $6a^2$. Thus $6a^2$ is a multiple of both $2a$ and $3a^2$, and we can write $6a^2=2a\cdot 3a$ and $6a^2=3a^2\cdot 2$. We write both $\frac{1}{2a}$ and $\frac{1}{3a^2}$ as equivalent expressions with the same denominator of $6a^2$ and add to obtain

$$\frac{1}{2a}+\frac{1}{3a^2}=\frac{1\cdot 3a}{2a\cdot 3a}+\frac{1\cdot 2}{3a^2\cdot 2}$$ Write each term as an equivalent expression with LCD $6a^2$

$$=\frac{3a}{6a^2}+\frac{2}{6a^2}$$ Simplify

$$=\frac{3a+2}{6a^2}$$ Add

(c) The *LCD* of the denominators $x-2$ and $x+1$ is $(x-2)(x+1)$. We write both $\frac{2}{x-2}$ and $\frac{3}{x+1}$ as equivalent expressions with the same denominator of $(x-2)(x+1)$, subtract and simplify to obtain

$$\frac{2}{x-2}-\frac{3}{x+1}=\frac{2(x+1)}{(x-2)(x+1)}-\frac{3(x-2)}{(x+1)(x-2)}$$ Write each term as an equivalent expression with LCD $(x-2)(x+1)$

$$=\frac{2(x+1)-3(x-2)}{(x-2)(x+1)}$$ Subtract

$$=\frac{2x+2-3x+6}{(x-2)(x+1)}$$ Expand numerator expressions

$$=\frac{-x+8}{(x-2)(x+1)}$$ Collect like terms

Example 7. Perform the addition or subtraction and simplify. Identify the *LCD* in each case.

(a) $2y+\frac{3}{y+1}$ **(b)** $\frac{1}{x^2+4x+4}-\frac{x+1}{x^2-4}$ **(c)** $\frac{3}{t}-\frac{2}{t+2}+\frac{4}{t^2+2t}$

Solution. **(a)** The term $2y$ is a rational expression with denominator 1. The *LCD* of the denominators 1 and $y+1$ is $y+1$. We write both $2y$ and $\dfrac{3}{y+1}$ as equivalent expressions with the same denominator of $y+1$ and add to obtain

$$\begin{aligned} 2y+\frac{3}{y+1} &= \frac{2y(y+1)}{y+1}+\frac{3}{y+1} && \text{Write each term as an equivalent expression with LCD } y+1 \\ &= \frac{2y^2+2y+3}{y+1} && \text{Expand } 2y(y+1) \text{ and add} \end{aligned}$$

(b) We must factor the denominator polynomials $x^2+4x+4=(x+2)^2$ and $x^2-4=(x-2)(x+2)$ to find their *LCD* $(x+2)^2(x-2)$. We write both $\dfrac{1}{x^2+4x+4}$ and $\dfrac{x+1}{x^2-4}$ as equivalent expressions with the same denominator of $(x+2)^2(x-2)$, subtract and simplify to obtain

$$\begin{aligned} \frac{1}{x^2+4x+4}-\frac{x+1}{x^2-4} &= \frac{1}{(x+2)^2}-\frac{x+1}{(x-2)(x+2)} && \text{Factor denominator polynomials} \\ &= \frac{x-2}{(x+2)^2(x-2)}-\frac{(x+1)(x+2)}{(x-2)(x+2)(x+2)} && \text{Write each term as an equivalent expression with LCD } (x+2)^2(x-2) \\ &= \frac{x-2-(x+1)(x+2)}{(x-2)(x+2)^2} && \text{Subtract} \\ &= \frac{x-2-(x^2+3x+2)}{(x-2)(x+2)^2} && \text{Multiply } (x+1)(x+2)=x^2+3x+2 \\ &= \frac{-x^2-2x-4}{(x-2)(x+2)^2} && \text{Collect like terms} \end{aligned}$$

(c) We factor $t^2+2t=t(t+2)$ and we see that the *LCD* of the denominators t, $t+2$ and $t(t+2)$ is $t(t+2)$. We write $\dfrac{3}{t}$, $\dfrac{2}{t+2}$ and $\dfrac{4}{t(t+2)}$ as equivalent expressions with the same denominator of $t(t+2)$, subtract and add the numerators, and simplify to obtain

$$\begin{aligned} \frac{3}{t}-\frac{2}{t+2}+\frac{4}{t^2+2t} &= \frac{3}{t}-\frac{2}{t+2}+\frac{4}{t(t+2)} && \text{Factor } t^2+2t=t(t+2) \\ &= \frac{3(t+2)}{t(t+2)}-\frac{2t}{(t+2)t}+\frac{4}{t(t+2)} && \text{Write each term as an equivalent expression with LCD } t(t+2) \\ &= \frac{3t+6-2t+4}{t(t+2)} && \text{Multiply } 3(t+2)=3t+6\text{; subtract and add} \\ &= \frac{t+10}{t(t+2)} && \text{Collect like terms} \end{aligned}$$

Simplifying Compound Fractions

A compound fraction is a rational expression whose numerator or denominator is also a rational expression. The next example shows how compound fractions can be simplified.

Example 8. Simplify the compound fraction.

(a) $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{1}{x}+\dfrac{1}{y}}$ **(b)** $\dfrac{\dfrac{1}{x}-\dfrac{1}{2}}{x-2}$

Solution. (a) We write both the numerator and denominator as a single fraction and simplify to obtain

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}=\frac{\frac{y^2-x^2}{x^2y^2}}{\frac{y+x}{xy}}$$ Add fractions in numerator and denominator

$$=\frac{y^2-x^2}{x^2y^2}\cdot\frac{xy}{y+x}$$ Invert and multiply

$$=\frac{(y-x)(y+x)\cdot xy}{x^2y^2\cdot(y+x)}$$ Factor

$$=\frac{(y-x)}{xy}$$ Simplify

(b) We write the numerator as a single fraction and simplify to obtain

$$\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\frac{\frac{2-x}{2x}}{x-2}$$ Add fractions in numerator and denominator

$$=\frac{2-x}{2x}\cdot\frac{1}{x-2}$$ Invert and multiply

$$=\frac{-(x-2)}{2x}\cdot\frac{1}{x-2}$$ Write $2-x=-(x-2)$

$$=-\frac{1}{2x}$$ Simplify