Section 1.1: Statements, Symbolic Representations, and Tautologies

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Abstract

In Section 1.1 we encounter the elements of logic: statements, connectives, tautologies, contradictions, etc., and create well-formed formulas (wffs - "whiffs") from these basic elements.

We discover the problems of transforming phrases in our language into the realm of logic: our language is ambiguous (perhaps we like that!), but that causes problems for us here.

A simple algorithm for detecting tautologies in the form of implications is described.

- **Statement/proposition**: a sentence possessing (or assigned) truth value (T or F; sometimes represented as 1 or 0).

  Exercise #1, p. 16: which of the following are statements?

  a. The moon is made of green cheese.
  b. He is certainly a tall man.
  c. Two is a prime number.
  d. The game will be over by 4:00. (What if we had said instead "Will the game be over soon?")
  e. Next year interest rates will rise.
  f. Next year interest rates will fall.
  g. $x^2 - 4 = 0$

  A couple of things we should observe in examining these examples:

  – "Truth is relative" (requires a **context** – the universe).
  – Variables come in several flavors. I mean, it's clear that "x" is a variable – it's our favorite variable! And it seems clear that "He" is a variable in "He is certainly a tall man." But is "tall" a variable? Is "moon" a variable?
  – English is a troublesome language (and this is just Chapter 1, exercise 1!).

- **Logical connectives** join statements into **formulas**, or arguments, or compound statements:
  - conjunction (symbolized by $\wedge$, "and")
  - disjunction (symbolized by $\vee$, "or")
  - implication (symbolized by $\rightarrow$, "implies")
  - equivalence (symbolized by $\longleftrightarrow$, "if and only if")
  - negation (symbolized by $'$ – "not" – which is a *unary* operation, in contrast to the other four which are *binary*)

  Note: These connectives are not independent - some of these may be derived from the others (Exercise #48 – one of your homework problems – shows that disjunction and negation suffice to write all the others, for example).

- **Well-formed formula** (wff - "whiff") is a compound statement made up of statements, logical connectives, and other wffs

  What makes one well-formed? There are just a few rules for creating wffs:

  a. All statements are wffs.
  b. The following are wffs, for any wffs A and B:
     a. $(A \vee B)$,
     b. $(A \wedge B)$,
     c. $(A \rightarrow B)$,
     d. $(A \longleftrightarrow B)$,
     e. A'

  This is a **recursive** method for defining wffs (we'll talk more about recursion soon....).

  Notice that all we are really saying is that wffs are just formed by applying the binary connectives and negation (the only unary connective). It's another way of saying that these connectives are indeed well-defined **functions**, defined on the set of all wffs: they operate on wffs, to produce new wffs.

  - **Order of precedence** in the "computation" of a wff:
    * parentheses
    * $'$
    * conjunction, disjunction
    * implication
    * equivalence

    Order of precedence helps us to simplify our lives: hence,

    $$A \wedge B \rightarrow C \text{ means } (A \wedge B) \rightarrow C$$

    So we can drop some parentheses, when they're properly handled by order of precedence.
  - The **main connective** is defined as the last to be applied.

- Truth Tables for the most common wffs:

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow B$	A'	B'	$A \leftrightarrow B$
T	T						
T	F						
F	T						
F	F						

Does the table for implication seem weird to you? It's by convention, but is also related to the fact that "from a falsehood anything can be proven". In the implication $A \rightarrow B$, A is the **antecedent**, and B is the **consequent**.

Bertrand Russell famously thus used the "fact" that "1=2" (a falsehood) to prove that he is the Pope - wouldn't the Pope be surprised!

Some English equivalents to implication are:

  - If A, then B.
  - A implies B.
  - A, therefore B.
  - A only if B.
  - A is a sufficient condition for B.
  - B follows from A.
  - B is a necessary condition for A.

Implication plays an especially important role among connectives, so learn it well!

Exercise #5 (p. 16) Find the antecedent and the consequent in each of the following statements:

a. Healthy plant growth follows from sufficient water.

b. Increased availability of information is a necessary condition for further technological advances.

c. Errors will be introduced only if there is a modification of the program.

d. Fuel savings implies good insulation or storm windows throughout.

Do you agree with the first one? Just because one writes a well-formed formula doesn't mean that it's reasonable (true)! Could it be false? Sometimes true, sometimes false?

Exercise #11ade (p. 17) - Negating implications and other wffs: Write the negation of each statement:

- a. If the food is good, then the service is excellent.
- d. Neither the food is good nor the service excellent.
- e. If the price is high, then the food is good and the service is excellent.

Some of the negations in Table 1.6 are suspect, IMHO. Can you guess why? (It "must be done with care"....)

Expressing the negation of a statement must be done with care, especially for a compound statement. Table 1.6 gives some examples.

TABLE 1.6

Statement	Correct Negation	Incorrect Negation
It will rain tomorrow.	It is false that it will rain tomorrow. It will not rain tomorrow.	
Peter is tall and thin.	It is false that Peter is tall and thin. Peter is not tall or he is not thin. Peter is short or fat.	Peter is short and fat. Too strong a statement. Peter fails to have both properties (tallness and thinness) but may still have one property.
The river is shallow or polluted.	It is false that the river is shallow or polluted. The river is neither shallow nor polluted. The river is deep and unpolluted.	The river is not shallow or not polluted. Too weak a statement. The river fails to have either property, not just fails to have one property.

- **Truth table** for a wff with n statement letters: 2^n rows

  The author makes an interesting observation about how we write the truth table 1: it can be constructed in such a way that it can be considered simply counting up using binary numbers, substituting "1" for "T", and "0" for "F". Our logical world is **binary**, so we should expect to see powers of two. We might also look at the trees for these logic tables 2, and relate them back to the entries in Pascal's (Yanghui's) triangle.

  Example: the table for implication above, which is a binary (2 statement letter) logical connective. Hence there are $2^2 = 4$ rows.

- **tautology**: wff which is always true (represented by 1).

TABLE 1.8		
A	**B**	**C**
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Figure 1:

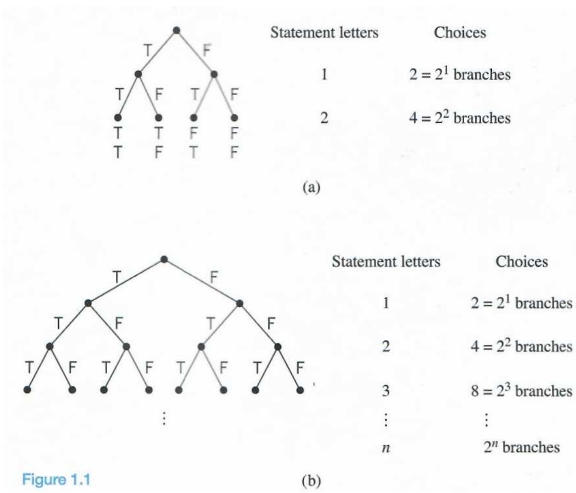

Figure 2:

- **contradiction**: wff which is always false (represented by 0).

- **equivalent wffs**: wffs A and B are equivalent (always have the same truth value), $A \iff B$, if the wff

$$A \longleftrightarrow B$$

is a tautology. (*How can we prove that?*)

Some tautological equivalences:

1a. $A \vee B \iff B \vee A$	1b. $A \wedge B \iff B \wedge A$	Commutative
2a. $(A \vee B) \vee C \iff A \vee (B \vee C)$	2b. $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$	Associative
3a. $A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$	3b. $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$	Distributive
4a. $A \vee 0 \iff A$	4b. $A \wedge 1 \iff A$	Identity
5a. $A \vee A' \iff 1$	5b. $A \wedge A' \iff 0$	Complement

Equivalent wffs will be useful when we are proving arguments, and want to replace complex wffs with simpler ones.

- **De Morgan's Laws** are two specific examples of equivalent wffs:
  - $(A \vee B)' \iff A' \wedge B'$
  - $(A \wedge B)' \iff A' \vee B'$

  Hence we claim that $(A \vee B)' \longleftrightarrow (A' \wedge B')$ is a tautology.

  Notice that the two formulas of De Morgan's Laws appear analogous ("dual"). In fact, one is the negation of the other.

Table 1: Exercise #20e: Verify by constructing a truth table that this example of De Morgan's law is a tautology: $(A \vee B)' \longleftrightarrow A' \wedge B'$.

A	B	$A \vee B$	$(A \vee B)'$	A'	B'	$A' \wedge B'$
T	T					
T	F					
F	T					
F	F					

Exercise #37

37. Rewrite the following statement form with a simplified conditional expression, where the function *odd*(n) returns true if n is odd.

```
if not((Value1 < Value2) or odd(Number))
or (not(Value1 < Value2) and odd(Number)) then
   statement1
else
   statement2
end if
```

- **Algorithm**: a set of instructions that can be mechanically executed in a finite amount of time in order to solve a problem.

  Often written out in **pseudocode**, the author provides us an example of an algorithm: TautologyTest, which is useful for determining whether or not an implication (that is, a wff where the main connective is implication) is, in fact, always true (a tautology). She proceeds by **contradiction** (one proof technique we'll study further in Chapter 2): assume that the implication $P \to Q$ is false. Then P must be true, and Q false (the only scenario which makes an implication false). One continues to break down each compound wff until one arrives at a contradiction. If one arrives at a contradiction, then the original wff (which we think of as a "theorem") is true.

Exercise 41c: Use TautologyTest to prove that this is a tautology: $(A \vee B) \wedge A' \to B$.

Building a truth table for the implication also constitutes an algorithm to test to see if it is true, but, although the truth table algorithm may be more powerful (as more general, working for all would-be tautologies), an algorithm like TautologyTest may be faster when applied to a particular implication.

We can use TautologyTest to check the conclusion of Lewis Carroll's syllogism:

a. Babies are crazy (people);
b. Crazy people should be in an asylum.
c. Therefore babies should be in an asylum.

Written as a tautology: $(B \to C) \wedge (C \to A) \to (B \to A)$.

Let's check. Assume that the wff is false: that is, assume that the lefthand side is true, but the righthand side is false.

But since $B \to A$ is false, we know that B is true, while A is false.

However, since both $B \to C$ and $C \to A$ are true, we must have C true (since B is true); then, since C is true, we must have A true to make the second implication true. But this is a contradiction.

Hence the proposed tautology is, in fact, a tautology.