Section 7.2: Euler Paths and Hamiltonian Circuits

March 31, 2025

Abstract

Graphs are useful for characterizing two interesting and important problems: the traveling salesman problem, and the highway inspector problem. The problem in each case is to traverse a network in an optimal way, with the focus on

- nodes (salesmen: Sir William Rowan Hamilton, 1805-1865)
- arcs (inspectors: Leonhard Euler, 1707-1783).

1 Euler Paths (Highway Inspector problem)

Definition: an **Euler Path** is a path in which each arc of the graph is used exactly once.

Euler got interested in these arcs when he encountered the Königsberg bridge problem in 1736 (p. 571); a game in which the object was to cross every bridge once without crossing any bridge twice. The old story is that Euler solved this problem by inventing and then using Graph Theory (disputed by our author – see the footnote on p. 571. You can decide for yourself, by reading Euler's original paper in translation).

From a letter of Leonhard Euler to Giovanni Marinoni, March 13, 1736:

A problem was posed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once.....This question is so banal, but seemed to be worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement.... (source)

The bridges are the arcs, and the land masses are nodes, turned into the graph of Figure 7.5, p. 571 (see Practice 9, below).

Example: Practice 7, p. 572 (unicursal/multicursal)

PRACTICE 7 Do Euler paths exist for either graph in Figure 7.6? (Use trial and error to answer. This is the old children's game of whether you can trace the whole graph without lifting your pencil and without retracing any arcs.)

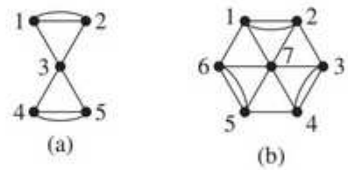


Figure 7.6

Theorem: in any graph, the number of odd nodes (nodes of odd degree) is even (the "hand-shaking theorem").

Outline of Gersting's proof:

a. Suppose that there are A arcs, and N nodes. Each arc contributes 2 ends; the number of ends is $2A$, and the degrees d_i satisfy

$$2A = \sum_{i=1}^{N} d_i$$

b.

$$2A - \sum_{i|d_i \ even} d_i = \sum_{i|d_i \ odd} d_i$$

and the left hand side is even (call it $2m$).

c. The sum of two odd degrees is even, so assume (we proceed by contradiction) that there is an odd number of odd nodes. We can pair up all but one (say $i = k$), and then

$$\sum_{i|d_i \ odd; \ i \neq k} d_i = 2n$$

d. From which we conclude that

$$2m - 2n = d_k$$

which means that d_k was, in fact, even; but this is a contradiction. Hence, the number of odd nodes is even.

Alternate proof: by induction on number of arcs, using cases.

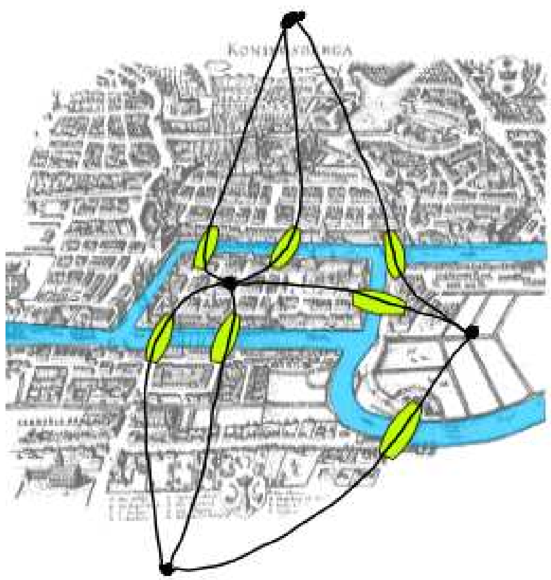
Theorem: an Euler path exists in a connected graph $\iff$ there are either two or zero odd nodes.

- Is this obvious? Why only two odd nodes?
- The "two odd node" case reduces to the even case (with one caveat, discussed below): start at one odd node, and trace a path to the other. Remove this subgraph, and what's left (and what might that be? What are the possibilities?) has even nodes only; so, since an Euler path exists for even noded graphs, we can reattach the pieces to form the original graph, with its Euler path.

  This assumes connectivity of the graph after removal of the path from odd to odd. Can you think of a case where you won't have connectivity?

Graph becomes disconnected if we trim out the arc between the odd nodes.

Example: Practice 9, p. 573 Is the Königsberg bridge walk possible?



No - four odd nodes.

(Click here for a modified problem that Euler considered.)

If you get frustrated trying to solve the Konigsberg bridge problem, you can always take the approach of spiked math...:).

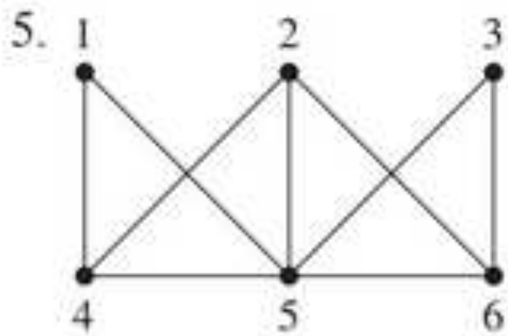
2 EulerPath Algorithm

The EulerPath algorithm (p. 574) makes use of the adjacency matrix representation of a graph to check for Euler paths. It simply counts up elements in a row i of the matrix (the degree of node i), and checks whether that's even or odd; if in the end there are not zero or two even nodes, there's no Euler path!

Example: Exercise 14, p. 578

14. Draw the adjacency matrix for the graph of Exercise 5. In applying algorithm *EulerPath*, what is the value of *total* after the fourth pass through the **while** loop?

5.



$$
\begin{array}{r c c}
 & & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\text{odd}=0 & 1 & \\
\text{odd}=1 & 2 & \\
\text{odd}=1 & 3 & \\
\text{odd}=2 & 4 & \\
\text{odd}=3 & 5 & \\
\text{\& done!} & 6 &
\end{array}
\quad
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
$$

Figure 1: Does our author's algorithm need to check $i = n$?

No need to check the last row: because of the hand-shaking theorem, the parity of the last row (odd or even) is known by the time we finish with the $n-1$ row.

EulerPath is $O(n^2)$, meaning that the number of operations in the worst case is on the order of n^2. So it's easy to have a computer determine whether there's an Euler path or not.

In contrast,

3 Hamiltonian Circuit Problem (the traveling salesman problem)

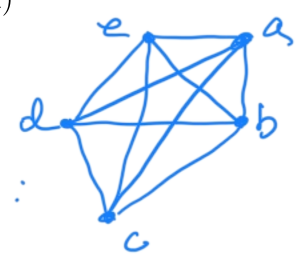
Definition: a Hamiltonian Circuit (or Cycle) is a cycle using every node of the graph (as a cycle, no node but the first is ever revisited, and that node is only visited at the beginning and end of the cycle).

"For example, consider a robot arm assigned to solder all the connections on a printed circuit board. The shortest tour that visits each solder point exactly once defines the most efficient path for the robot. A similar application arises in minimizing the amount of time taken by a graphics plotter to draw a given figure." (the Travelling Salesman Problem)

This is a hard problem!

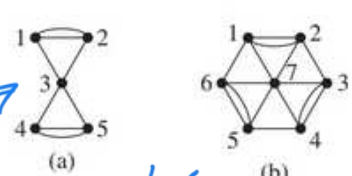
An example is a complete graph, like K_5: there is a path from each node to every other node, so no matter where you start, you can trace a cycle through every node – and in any order!).

e.g. K_5:



Example: Practice 11, p. 576

PRACTICE 11 Do Hamiltonian circuits exist for the graphs of Figure 7.6? (Use trial and error to answer.)

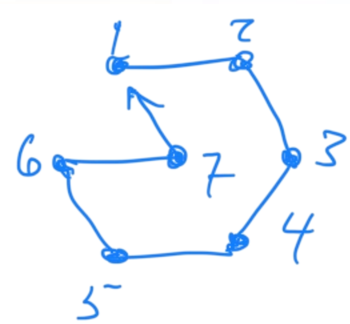


(a) (b)

Figure 7.6

choke point / bottleneck

No sweat:



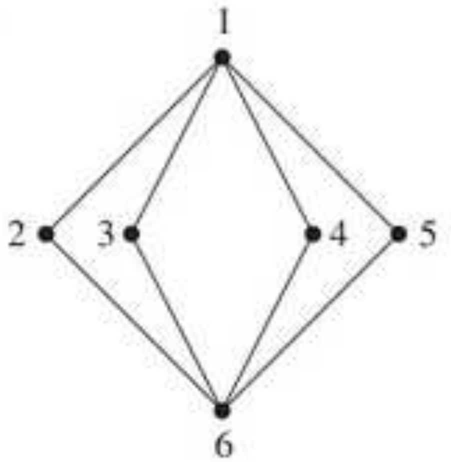
Not possible – must revisit 3.

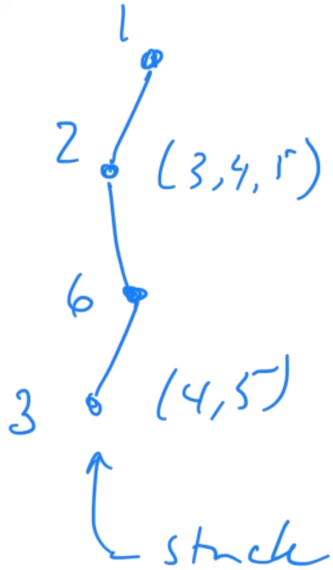
Example: Exercise 21, p. 579 (using trees, symmetry, and exhaustion!)

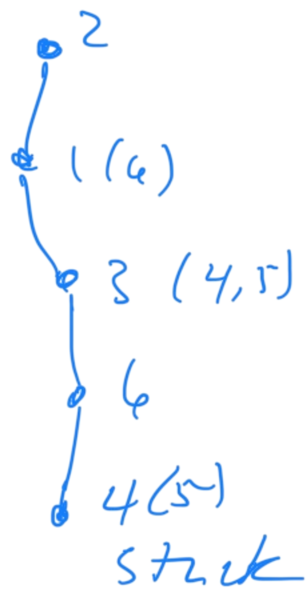
For Exercises 21–28, decide by trial and error whether Hamiltonian circuits exist for the graphs of the given exercise. If so, list the nodes in such a cycle.

21. Exercise 3 23. Exercise 5 25. Exercise 7 27. Exercise 9

3.







By symmetry, we're done.

Unfortunately, there's no nice HamiltonCircuit algorithm for determining when there is a Hamiltonian circuit (only very grungy, computationally intensive ones!).