Section 4.1: Sets

March 17, 2025

Abstract

This section, the only section we consider from Chapter 4, gives us some basic vocabulary and notions of sets that we will need when we get to Boolean algebras later. We observe that the rules satisfied by the binary operations of "intersection" and "union" from set theory are essentially the same as the rules of the binary connectives $\wedge$ and $\vee$ of propositional logic.

One of the most important ideas in this section is that of the "power set" – the set of all subsets of a set.

We also **prove** that there are infinitely many different sizes of infinite sets – did you know that?

1 Notation

A **set** (call it A) is loosely a collection of objects within some **universe**; the objects are called the **elements** of A.

Capital letters denote sets, and $\in$ denotes membership in a set, so that $x \in A$ means that x is a member (or element) of a set, and $x \notin A$ means that x is **not** a member.

Sets are unordered: the order in which the elements are listed is unimportant.

We can use predicate logic to determine (or even define) when two sets are **equal**:

$$A = B \iff (\forall x)[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

The notation for a set whose elements are characterized by possessing property P is

$$S = \{x | P(x)\}$$

and is read "S is the set of all x such that $P(x)$"

One curiously useful set is the **empty** set, denoted $\emptyset$ or $\{\}$.

Some important sets of numbers:

- $\mathbb{N}$ Natural numbers – although our author throws in 0, argh!
- $\mathbb{Z}$ Integers – positive and negative natural numbers, plus 0
- $\mathbb{Q}$ Rational numbers – reals expressible as ratios of integers
- $\mathbb{I}$ Irrational numbers – reals **not** expressible as ratios of integers
- $\mathbb{R}$ Real numbers – the continuum of numbers on the real number line
- $\mathbb{C}$ Complex numbers – including the important number $i = \sqrt{-1}$

I was always taught that the natural numbers start from 1. In particular, 0 is not at all "natural" – it must have required quite a stretch for a civilization to realize that they needed **a symbol for nothing**!

Example: Practice 3, p. 224. Describe each set:

(a) $A = \{x | x \in \mathbb{N} \wedge (\forall y)(y \in \{2,3,4,5\} \rightarrow x \geq y)\}$ $= \{x \mid x \in \mathbb{N} \wedge x \geq 5\}$

(b) $B = \{x | (\exists y)(\exists z)(y \in \{1,2\} \wedge z \in \{2,3\} \wedge x = y + z)\}$ $= \{3, 4, 5\}$

2 Relationships between Sets

A is a **subset** of B, denoted $A \subseteq B$, if

$$(\forall x)(x \in A \rightarrow x \in B)$$

and A is a **proper subset** of B, denoted $A \subset B$, if

$$(\forall x)(x \in A \rightarrow x \in B) \wedge (\exists x)(x \notin A \wedge x \in B)$$

Example: Practice 6, p. 225

PRACTICE 6 Let

$$A = \{x | x \in \mathbb{N} \text{ and } x \geq 5\}$$
$$B = \{10, 12, 16, 20\}$$
$$C = \{x | (\exists y)(y \in \mathbb{N} \text{ and } x = 2y)\}$$

Which of the following statements are true?

a. $B \subseteq C$ ✓
b. $B \subset A$ ✓
c. $A \subseteq C$ ✗
d. $26 \in C$ ✓
e. $\{11, 12, 13\} \subseteq A$ ✓
f. $\{11, 12, 13\} \subset C$ ✗
g. $\{12\} \in B$ ✗ $12 \in B$,
h. $\{12\} \subseteq B$ ✓
i. $\{x | x \in \mathbb{N} \text{ and } x < 20\} \not\subseteq B$ ✗
j. $5 \subseteq A$ ✗ $\{5\} \subseteq A$
k. $\{\varnothing\} \subseteq B$ ✗ $\varnothing \subseteq B$
l. $\varnothing \notin A$ ✓

Theorem:

$$A = B \iff A \subseteq B \wedge B \subseteq A$$

3 Sets of Sets

Power Set: Given set S, the power set of S, denoted $\wp(S)$, is the set of all subsets of S. (Note that S and $\oslash$ are themselves <u>elements</u> of the power set of S.)

Example: How big is the power set of a given set? (Practice 8 and 9, p. 227; power set for $A = \{1, 2, 3\}$, and how many elements in the power set more generally?)

$\wp(A) = \{ \phi,$
$\{1\}, \{2\}, \{3\},$
$\{2,3\}, \{1,3\}, \{1,2\},$
$\{1,2,3\} \}$

1
3
3
1

$Card(\phi) = 0$
$Card(\wp(\phi)) = 2^0$
$= 1$:
$\{\phi\} = \wp(\phi)$

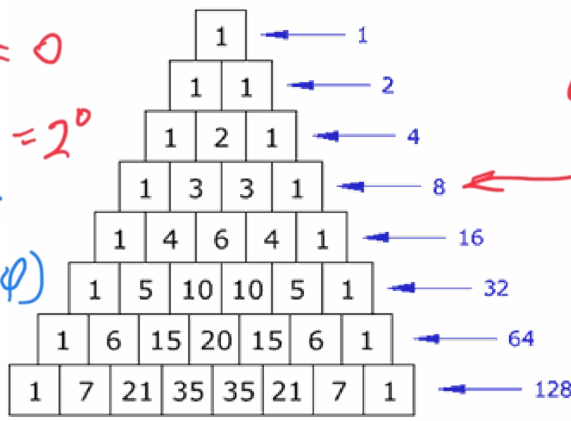


Figure 1: Pascal's Triangle (aka binomial coefficients)

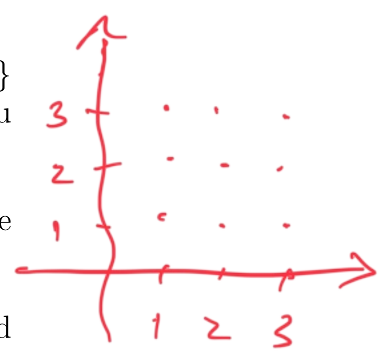
$Card(A) = 3$
$Card(\wp(A)) = 2^3 = 8$

Pascal's triangle gives you the breakdown on the number of various sized subsets you can create from a set of a given size. The line number n in the triangle (indexed from 0) tells you the size (n) of the underlying set, and the total across the row tells you just how many subsets there are (2^n). This is called the **cardinality** of the set: $Card(\wp(S)) = 2^n$ – its size. But cardinality is used even for infinite sets.

$Card(A) = n \rightarrow$
$Card(\wp(A)) = 2^n$

4 Binary and Unary Operations

We can create **ordered pairs** of elements of a set. From $A = \{1, 3, 4\}$ we can create the ordered pairs $(1, 3)$ and $(3, 3)$, for example. As you can tell from the name, the order of the elements is important!



Question: How many distinctly different ordered pairs are there if we have a set with n elements?

$Card(A) = n \rightarrow$
n^2 ordered pairs

Definition: $\circ$ is a **binary operation** on a set S if for every ordered pair (x, y) of elements of S, $x \circ y$ exists, is unique, and is a member of S.

Definition: $\circ$ is **well-defined** if $x \circ y$ exists and is unique.

Definition: $\circ$ is **closed** if $x \circ y \in S$.

Three ways to fail to be a binary operation on S:

(a) there are pairs for which $x \circ y$ fails to exist;

(b) there are pairs for which $x \circ y$ gives multiple different results;

(c) there are pairs for which $x \circ y$ doesn't belong to S.

Definition: a **unary operation** on a set S associates with every element x of S a unique element of S.

Example: Practice 12, p. 230

PRACTICE 12 Which of the following candidates are neither binary nor unary operations on the given sets? Why not?

a. $x \circ y = x \div y$; S = set of all positive integers ✗
b. $x \circ y = x \div y$; S = set of all positive rational numbers ✓
c. $x \circ y = x^y$; $S = \mathbb{R}$ ✗
d. $x \circ y$ = maximum of x and y; $S = \mathbb{N}$ ✓
e. $x^{\#} = \sqrt{x}$; S = set of all positive real numbers ✗
f. $x^{\#}$ = solution to equation $(x^{\#})^2 = x$; $S = \mathbb{C}$ ■

5 Operations on Sets

Given a set S of elements of interest (the **universal set**), we may want to operate on various subsets of S (that is, elements of $\wp(S)$). For example,

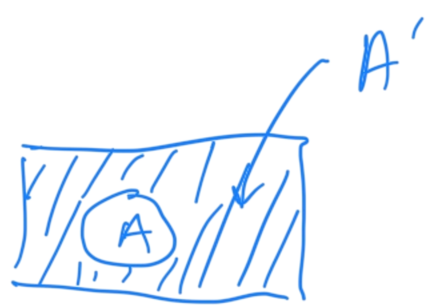
Definition: Let $A, B \in \wp(S)$. The **union** of A and B, denoted $A \cup B$, is given by $\{x | x \in A \vee x \in B\}$. The **intersection** of A and B, denoted $A \cap B$, is given by $\{x | x \in A \wedge x \in B\}$.

These are examples of binary operations on the power set of a set.

Definition: For a set $A \in \wp(S)$, the **complement** of A, denoted A', is $\{x | x \in S \wedge x \notin A\}$.

Definition: For sets A and $B \in \wp(S)$, the **set-difference** of A and B, denoted $A - B$, is given by $\{x | x \in A \wedge x \notin B\}$.

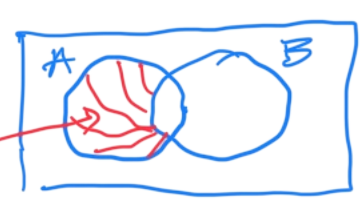
Venn Diagrams are useful tools for visualizing the notions of union and intersection. The diagrams in Figures 4.1 and 4.2 (p. 231) illustrate these notions "pictorially":



Examples:

(a) Practice 14, p. 232: illustrate A' using a Venn Diagram.

(b) Practice 15, p. 232: illustrate $A - B$ using a Venn Diagram.



$A - B = A \cap B'$

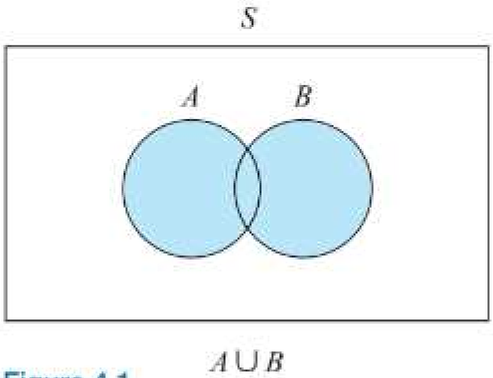


Figure 4.1

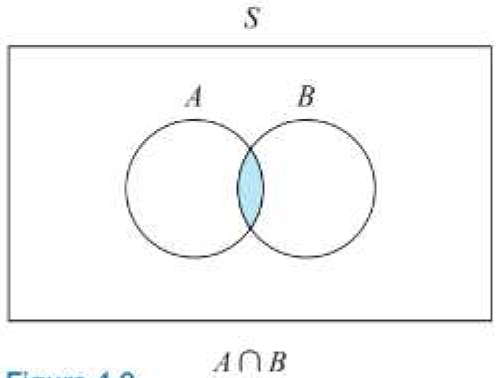
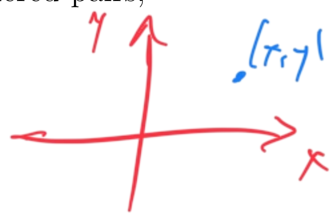


Figure 4.2

Definition: For set $A, B \in \wp(S)$, the **Cartesian product (cross product)** of A and B, denoted $A \times B$, is the set of all ordered pairs, and is given by

$$A \times B = \{(x, y) | x \in A \wedge y \in B\}.$$

$A = \Re \quad B = \Re$



The most famous Cartesian cross-product: the Cartesian coordinate system.

6 Set Identities

We will encounter the following "Set identities" later in the context of "Boolean algebras":

1a. $A \cup B = B \cup A$	1b. $A \cap B = B \cap A$	commutative property
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$	associative property
3a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	3b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	distributive property
4a. $A \cup \oslash = A$	4b. $A \cap S = A$	identity property
5a. $A \cup A' = S$	5b. $A \cap A' = \oslash$	complement property

Notice again the "dual" nature of the properties: it seems that the operations of $\cup$ and $\cap$ have a lot in common!

Question: What correspondence do you observe between these identities and those of wffs with the logical connective $\wedge$ and $\vee$?

7 Countable and Uncountable Sets

As an interesting application of set theory, we will now demonstrate that **there are infinitely many sizes of infinity**.

The natural numbers comprise the smallest infinity, a **denumerable** or **countable** infinity.

We prove that two sets are of equal size (even if infinite!) by creating a **one-to-one correspondence** between the two sets: $f : A \to B$. If such a correspondence exists, then the two sets have the same size.

$\vdots$

$\text{card}(P(P(P(S)))) > \text{card}(P(P(S))) >$

$\boxed{\text{card}(P(S)) > \text{card}(S)}$

for all sets S — even infinite sets.

By **one-to-one correspondence** we mean that each element of each set has a unique partner (no member of either set is "left behind"). I imagine a dance, where all elements of both sets are happily dancing with their special partners. Such a partnership is actually a one-to-one and **onto** mapping: not only does each x have a unique partner y (one-to-one), but *vice versa* (so every element y is a partner).

E - even ℕ

$n \in \mathbb{N}$; assign $2n$ as its partner in E: $f: \mathbb{N} \to E$ by $n \to 2n$

$\text{Card}(\mathbb{N}) = \text{Card}(E)$

every natural number n is at the dance, with $2n$

every even is at the dance, with its partner.

Example: The even natural numbers E are the same size as the natural numbers, as shown by the one-to-one correspondence

$$f : \mathbb{N} \to E \text{ given by } n \longleftrightarrow 2n$$

(Notice that each element of E is a partner of a natural number).

Theorem: the rational numbers (ratios of integers) are countable.

Theorem (Cantor's diagonalization argument, Example 23, p. 238): the real numbers are **not** countable.

Theorem: the power set of a set S is always larger than S (punch line: there is always a bigger infinity than the one you already have).

Proof: By contradiction. Consider $f : S \to \wp(S)$ a one-to-one correspondence between S and $\wp(S)$. That is, every element of S is partnered with a unique element of $\wp(S)$ (and *vice versa*). (We will show that this is impossible.)

$\overset{\in S}{x} \to \underset{\in \wp(S)}{f(x)}$

Denote by $f(S)$ the set of subsets that are the images of all the elements of S: $f(S) \equiv \{f(x)|x \in S\}$. Then we have asserted that $f(S) = \wp(S)$ – that is, that every subset of S is the image of some element of S.

We'll show that there aren't enough elements of S to pair up with all the elements of $\wp(S)$.

However, consider the subset of S given by

$$A = \{x \in S | x \notin f(x)\}$$

But $A \notin f(S)$ (because it's different from every element $f(x)$ of $f(S)$), by design; and yet $A \in \wp(S)$. This is a contradiction: we asserted that the mapping was one-to-one – i.e., that $f(S) = \wp(S)$.

Just to try to make the nature of the set A a little clearer, here's the purported one-to-one mapping by f:

$$\begin{array}{ccc} \underline{S} & & \underline{\wp(S)} \\ x_1 & \to & B_1 = f(x_1) \\ x & \to & B = f(x) \\ x^* & \to & B^* = f(x^*) \\ \vdots & & \vdots \end{array}$$

$x_1 \in B_1$; then $x_1 \notin A$
$x \notin B$; then $x \in A$
$x^* \notin B^*$; then $x^* \in A$

But $A = \{x \in S | x \notin f(x)\}$ is different from each of the sets on the right-hand side, by construction: for example, if $x_1 \in B_1$, then A rejects it (and hence is different from B_1); if $x \notin B$, then A accepts it (and hence is different from B); if $x^* \notin B^*$, then we take $x^* \in A$ (and hence A is different from B^*); and so on. It's the same argument as Cantor's diagonalization argument, on steroids....

A is different from every dance partner. ∴ there is a member of the $\wp(S)$, A, that's not at the dance. The power set is just too big....!