Section 2.2: Induction

February 5, 2025

Abstract

In this section we investigate a powerful form of proof called **induction**. This is useful for demonstrating that a property, call it $P(n)$, holds **for all** natural numbers (integers n greater than or equal to 1).

Actually, the "1" above is not essential: any "base integer" will do (like 0, for example: it really only matters that there be a beginning – a "ground floor", or "base case", or "anchor").

1 Induction

Induction is a very beautiful and somewhat subtle method of proof: the objective is to demonstrate a property associated with the natural numbers[1], $\mathbb{N} = 1, 2, 3, \ldots, n, \ldots$. As a typical example, consider a theorem of the following type (which we might call "Gauss's theorem," hypothesized when he was seven or so):

Prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

An induction proof goes something like this:

- We'll show that it's true for the first case (usually $k = 1$, called the base case). While the first case is often $k = 1$, this isn't mandatory: we simply need to be sure that there **is** a first case for which the property is true. $k = 0$ is another popular choice....
- Then we'll show that, if the property is true for the k^{th} case, then it's true for the $(k+1)^{th}$ case (the inductive step).
- Then we'll put them together: if it's true for 1, then it's true for 2; if it's true for 2, then it's true for 3; .... "to infinity, and beyond!" Or "up the ladder", as our author would say.

Imagine dominoes falling. That's what it's like.

The most commonly used form of the principle of induction is expressed as follows:

First Principle of Mathematical Induction:

$$\left.\begin{array}{l} 1.\ P(1) \text{ is true} \\ 2.\ (\forall k)[P(k) \text{ true } \rightarrow P(k+1) \text{ true }] \end{array}\right\} \rightarrow P(n) \text{ true for all positive integers } n$$

[1](or another subset of the integers with a "least" element)

or, more succinctly,

$$P(1) \wedge (\forall k)[P(k) \to P(k+1)] \to (\forall n)P(n)$$

where the domain of the interpretation is the natural numbers. This is just *modus ponens* applied over and over again. Put *modus ponens* into an infinite loop, because we want it to run off to infinity! This might be the first infinite loop you've ever liked....

1. $P(1)$
2. $(\forall k)(P(k) \to P(k+1))$
3. $P(1) \to P(2)$ ui
4. $P(2)$
5. $P(2) \to P(3)$ ui
6. $P(3)$

etc.

$\forall n$!

Vocabulary:

- **inductive hypothesis**: $P(k)$. This is an **assertion**, about k.
- **basis step** ("base case", "anchor"): establish $P(1)$.
- **inductive step** (implication): $P(k) \to P(k+1)$. For this step, you will assume $P(k)$ is true, and then show that $P(k+1)$ follows. **Attention**: one almost never writes "$P(k+1) = ...$". $P(k+1)$ is an **assertion**, to be established; it is **not** a quantity!

Example: Dominoes

Example: Practice 7 (or "Gauss's theorem"), p. 115 Prove that, for any natural number n, $\underbrace{1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}}$.

$P(n)$:

① (Anchor) - Check $P(1)$

$$1 \overset{?}{=} \frac{1(1+1)}{2} = 1 \checkmark$$

② (Inductive) Assume $P(k)$. Consider $P(k+1)$:

$$\underbrace{1+2+\ldots+k}_{\frac{k(k+1)}{2}} + (k+1) = \frac{k(k+1)}{2} + (k+1) = \left(\frac{k}{2}+1\right)(k+1)$$

$$= \left(\frac{k+2}{2}\right)(k+1)$$

$$= \left(\frac{(k+1)+1}{2}\right)(k+1)$$

$$= \frac{(k+1)((k+1)+1)}{2} \checkmark$$

$\therefore (\forall n)P(n)$ by the 1st principle of induction

$P(k): k! \geq 2^{k-1}$

Example: Exercise 39, p. 125 Prove that $n! \geq 2^{n-1}$ for $n \geq 1$.

$P(n): n! \geq 2^{n-1}$

$P(k+1): (k+1)! \geq 2^{(k+1)-1}$

Check $P(1)$: $1! = 1 \overset{?}{\geq} 2^{1-1} = 2^0 = 1$ ✓

Assume $P(k)$, & consider LHS of $P(k+1)$:

$$(k+1)! = (k+1)k! \geq (k+1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^{(k+1)-1} \checkmark$$

used $P(k)$

$k+1 \geq 2$

This is important for computer scientists especially (and all scientists, really): it means that factorial functions grow even faster than exponential functions.

$$(k+1)! = (k+1)k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1$$

A second (and seemingly more powerful) form of induction is given by the **Second Principle of Mathematical Induction**:

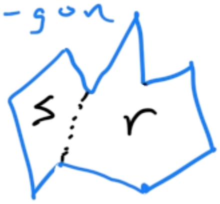
$$\left.\begin{array}{l} 1.\ P(1) \text{ is true} \\ 2.\ (\forall k)[P(r) \text{ true for all } r, \\ \qquad 1 \le r \le k \rightarrow P(k+1) \text{ true }] \end{array}\right\} \rightarrow P(n) \text{ true for all positive integers } n$$

This principle is useful when we cannot deduce $P(k+1)$ from $P(k)$ (for k alone), but we can deduce $P(k+1)$ from all preceeding cases, beginning with the base case.

P(3): sum of interior angles of a triangle is 180° ✓ $= (3-2)180$

Example: Exercise 72b, p. 127. Use the second principle of induction to prove that the sum of the interior angles of an n-sided simple closed polygon is $(n-2)180^o$ for all $n \geq 3$.

k+1-gon



Base case: $n=3$ (a triangle)

Assume $P(r)$ true for all $3 \le r \le k$.

Divide a $(k+1)$-gon into gons, s-gon & r-gon.

$P(s)$ & $P(r)$ are true by assumption. The sum of the interior angles hasn't changed: $P(s)$: $(s-2)180$ $P(r)$: $(r-2)180$

$\therefore$ sum for k+1gon is $(s-2)180 + (r-2)180 = (\underbrace{s+r-2}_{} - 2)180$

$= ((k+1) - 2)180$ ✓

In spite of appearances, these two forms (or principles) of mathematical induction are equivalent; furthermore they are also equivalent to the **Principle of Well-Ordering**, which states that every collection of positive integers that contains any members at all has a smallest member.

FPI → Well-ordering

Example: Prove that the first principle of induction implies well-ordering. We'll do this by combining contradiction and induction.

Assume FPI. Assume ∃ a non-empty set S of pos. integers that contains no smallest member (we'll contradict this!). Since S is non-empty, it contains some positive integer (call it a).

$P(n)$: every member of S is greater than n.

Base case: $P(1)$ is true, as otherwise $n=1$ is in S, & S has a least element. ✓

Inductive step: Assume $P(k)$: every element of S is greater than k. Then if $P(k+1)$ is false, then $k+1$ is in S, making it the smallest member, a contradiction. Hence $P(k+1)$ ✓ $P(k) \Rightarrow P(k+1)$, & $(\forall n)P(n)$.

A Couple of Fun Examples:

But $P(a)$ is false! Thus we arrive at a contradiction, & we conclude that every set of positive integers has a least element.

(well-ordering)

(a) All natural numbers are interesting.

(b) The prisoner's last request (finite backwards induction!)

A prisoner, condemned to die by the Sultan of an antique land, made a plaintive request: "Please Sultan, if you would only grant me two favors: one, that you have me executed in the month of January (next month), and two, that you don't allow me to know the day of my death until 10 a.m. of the day upon which I am to die." The Sultan, being a merciful man, granted these requests, whereupon the prisoner demonstrated that it was impossible to execute him subject to these conditions. How?[2]

(c) Now that we understand induction, let's use it to prove an amazing fact: All horses are the same color.

Proof: By induction, on the number of horses.

Base case: 1 horse. No problem! Same color.

Inductive step: we'll show that if it is true for any group of N horses, that all have the same color, then it is true for any group of $N+1$ horses.

Well, given any set of $N+1$ horses, if you exclude the last horse, you get a set of N horses. By the inductive step these N horses all have the same color. But by excluding the first horse in the pack of $N+1$ horses, you can conclude that the last N horses also have the same color. Therefore all $N+1$ horses have the same color.

QED – or have we?

[2] What do you think happened to the prisoner?