October 25, 2000

# The Transition to Calculus [1]

## 1 Introduction

By the end of the $16^{th}$ century the European algebraists had achieved about as much as possible following the Islamic Tradition. They were expert at algebraic manipulation, could solve cubics and quartics, had developed an effective syncopated notation, though they had yet no notation for arbitrary coefficients. This compelled them to explain methods by example. Formulas for the solution of equations did not yet exist.

Another trend that began during this time was the reconstruction of all of Greek mathematics. The basic library (Euclid, Ptolemy, etc.) had been translated earlier.

The Italian geometer Federigo Commandino (1509-1575) stands out here, as he prepared Latin translations of all known works of Archimedes, Apollonius, Pappus, Aristarchus, Autocyclus, Heron, and others. He was able to correct numerous errors that had crept in over the centuries of copying and translating.

The desire here, beside the link with the past, was to determine the Greek methods. Most of the Greek manuscripts were models of synthetic analysis (axiom, theorem, proof,...) with little clue to methods of discovery. The work of Pappus (Book VII) was of some help here.

## 2 Francois Viéte

Francois Viéte (1540 - 1603), a Frenchman, was an attorney and public administrator at the court of King Henry II. His father, Etienne Viète, was also an attorney in Fontenay and a notary in Le Busseau. His grandfather was a merchant in the village of Foussay in Lower Poitou. The Viète family can be placed among the most distinguished in Fontenary. His two brothers both had distinguished positions. Viète's first scientific work was his set of lectures to Catherine Parthenay of which only *Principes des cosmographie* survives. This work introduced his student to the sphere, elements of geography, and elements of astronomy.

His mathematical works are closely related to his cosmology and works in astronomy. In 1571 he published *Canon mathematicus* which was to serve as the trigonometric introduction to his *Harmonicon coeleste* which was never published.

Twenty years later he published *In artem analyticum isagoge* which was the earliest work on symbolic algebra.

In 1592 he began his dispute with Scaliger over his purported solutions to the classical problems with ruler and compass.

Viète was involved in the calendar reform but rejected Clavius'[2] and in 1602 published a vehement attack against the calendar reform and Clavius. The dispute ended only with Viète's death at the beginning of 1603.

He one of the first men of talent who attempted to identify with the new Greek analysis. Viète wrote several treatises, together called *The Analytic Art*. In them he reformulated the study of algebra for the solution of equations by focusing on their structure. Thus he developed the earliest *articulated theory* of equations.

In *The Analytic Art* he distinguished three types of analysis, after

[2]Christopher Clavius (1537 - 1612), an Italian Jesuit astronomer, wrote the papal bull which appeared in March 1582 on calendar reform for Pope Gregory XIII. In 1574 Clavius, also a geometer, showed that the parallel postulate is equivalent to asserting that the line equidistant from a straight line is a straight line.

Pappus' two (theorematic, problematic). His trichotomy was:

- **zetetic analysis** – transformation of a problem to an equation.
- **poristic analysis** – exploring the truth of a conjecture by symbol manipulation.
- **exegetics** – the art of solving the equation found by zetetic analysis.

Features of *The Analytic Art*:

- – Fully symbolic for **knowns** (consonants) and unknowns (vowels).
- – Uses words and abbreviations for powers àla Bombelli and Chuquet. He writes $A$ quadratum for $A^2$, $B$ cubus for $B^3$, et al.
- – Follows the German in using $+$ and $-$.
- – For multiplication he uses the word **in**:

$$\frac{A \text{ in } B}{C \text{ cubus}} = \frac{AB}{C^3}.$$

- – For square root he uses $L$: $L64 = 8$.
- – As others Vieta insisted on homogeneity: so in writing $x^3 + ax$, $a$ must be a plane number so $ax$ is solid.
- – Advocated the use of decimal numbers using the , as separatrix and an underbar for the fractional part

$$141,421,\underline{356,24} = \sqrt{2} \times 100,000.$$

[The decimal point, . , was suggested by G.A. Magini (1555-1617).]

Viète's symbolic operations

- – Multiplies $(A + B)$ and $(A - B)$ to get $A^2 - B^2$:

$$(A + B) \text{ in } (A - B) \text{ equalx } A \text{ quadratum} - B \text{ quadratum}$$

- – and notes that

$$(A+B)^2 - (A-B)^2 = 4AB.$$

- – He also writes out products such as

$$\begin{aligned} (A-B)(A^2+AB+B^2) &= A^3 - B^3 \\ (A-B)(A^3+A^2B+AB^2+B^3) &= A^4 - B^4. \end{aligned}$$

- – He combines algebraic manipulation with trigonometry. Assume $B^2 + D^2 = X^2$ and $F^2 + G^2 = Y^2$: Then another right triangle can be constructed by reason of the formula:

$$(BG + DF)^2 + (DG - BF)^2 = (B^2 + D^2)(F^2 + G^2).$$

Viète's Theory of Equations

- – He replaced the 13 cases of cubics, as described by Cardano and Bombelli to a single transformation wherein the quadratic term is missing

$$ax^3 + bx^2 + cx + d = 0 \to y^3 + \gamma y + \delta = 0.$$

- – He shows for the form

$$x^3 - b^2x = b^2d$$

comes from the existence of the form continued proportionals

$$b : x = x : y = y : (x + d)$$

i.e.

$$\begin{aligned} \frac{b}{x} &= \frac{x}{y} \to x^2 = by \to x^4 = b^2y^2 \\ \frac{x}{y} &= \frac{y}{x+d} \to y^2 = x(x+d). \end{aligned}$$

Now substitute and cancel. From this he solves the cubic $x^3 - 64x = 960$.

- – Viète also used the cubic, in a different form, and trigonometric identities to generate other solution methods.

- – Viète studied the relation of **roots** to **coefficients**: If $x_1$ and $x_2$ are roots to $x^3 + b = 3ax$, then

$$\begin{aligned} 3a &= x_1^2 + x_1 x_2 + x_2^2 \\ b &= x_1 x_2^2 + x_1^2 x_2. \end{aligned}$$

- – In trigonometry, Viète used clever manipulations to arrive at multiple angle formulas equivalent to the modern form

$$\begin{aligned} \cos nx &= \cos^n x - \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} x \sin^2 x \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^{n-4} x \sin^4 x \ldots \end{aligned}$$

  and a corresponding formula for $\sin nx$.

- As an anecdote, Viète used this formula to solve the 45th degree equation

$$x^{45} - 45x^{43} + 945x^{41} \cdots + 45x = K,$$

  issued as a challenge by **Adriaen van Roomen** (1561-1615), a Belgian mathematician. Viète noticed that this equation arises when $K = \sin 45\theta$ and one wishes to express it in terms of $x = 2\sin\theta$. Apparently van Roomen was impressed, so much so that he paid a visit to Viète.

Viète also gave us our first closed form for $\pi$, from the formula

$$\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots}$$

using trigonometry of inscribed figures. This can be derived starting with a inscribed square of area $a_1$, and proceeding to the circle by success doubling of the number of sides. Generate areas $a_1, a_2, a_3, \ldots$ . Now show $a_{n+1} = a_n \sec(\pi/2^{n+1})$.

Algorithm: Let $a_2 = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}$. For $n = 3, 4, \ldots N$ do $a_n = \sqrt{\frac{1}{2} + \frac{1}{2}a_{n-1}}$. Define $p = 2\sqrt{2} \Big/ \prod_{j=2}^{N} a_n$.

## 3 Simon Stevin


Simon Stevin (1548-1620), a Belgian, was from a poor family and was in fact illegitimate. Though he was raised by his mother, his father, likely an artisan by trade, was Antheunis Stevin. In 1581, having worked as a bookkeeper and clerk in Antwerp he moved to Leiden (Netherlands) and enrolled in the Latin school. Later he took courses at the university. Though he studied there for seven years up until 1590, there is no evidence of his graduation.

He eventually rose to become commissioner of public works and quartermaster general of the army under Prince Maurice of Nassau. As an engineer, his principal profession, he designed a series of sluices used to flood certain areas and drive off any enemy, an important defense of Holland.

He published extensively on mathematics, engineering (both military and civil), hydrodynamics and mechanics. His work on mathematics included practical surveying, with descriptions of instruments for that purpose. Stevin's contributions to mathematics include:

- (1) Excellent mathematical notation for decimal fractions – for which he was a passionate advocate.
- (2) Replacement of the sexagesimal system by the decimal system.
- (3) Introduction of double-entry bookkeeping in the Netherlands – after Pacioli.
- (4) Computation of quantities that required limiting type processes.
- (5) Diminishing the Aristotelian distinction between number and magnitude.
- (6) Opposed the exclusive use of Latin in scientific writing; after 1583 he published only in Dutch.

In the history of science Stevins name is also prominent. He and a friend dropped two spheres of lead, one ten times the weight of the other, 30 feet onto a board. They noticed the sounds of striking to be almost simultaneous. (1586) Galileo's *later* similar experiment got more press.

### 3.1 Decimal Fractions.

His work on decimal fractions was contained in his 1585 book, *De Thiende* (*The Art of Tenths*). It was widely read and translated. His goal in this book was as a teacher, to explain in full and elementary detail how to use the decimals. His idea was to indicate the power of ten associated with each decimal digit. We write for the approximation of $\pi$

$$\pi \doteq 3 \text{ⓞ} 1 \text{①} 4 \text{②} 1 \text{③} 6 \text{④}$$

or

$$\pi \doteq \begin{matrix} \text{ⓞ} & \text{①} & \text{②} & \text{③} & \text{④} \\ 3 & 1 & 4 & 1 & 6 \end{matrix}$$

Similarly,

$$8\frac{937}{1000} = 8 \text{ⓞ} 9 \text{①} 3 \text{②} 7 \text{③}$$

In the second part of his book he shows how to calculate with these numbers.

His notation was streamlined to the modern form only 30 years later in the 1616 English translation of Napier's *Descriptio*, complete with the decimal point.

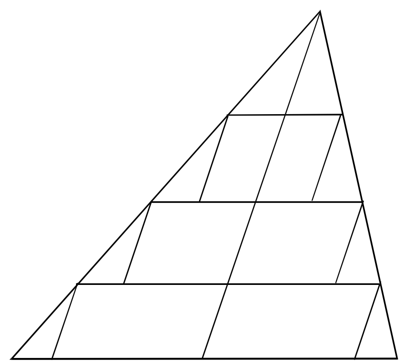
Stevin also advocated the use of base 10 for units of all quantities. It would be more than 200 years before the French introduced the metric system. His main intent was to provide enough understanding to make of mathematics a serviceable tool.

### 3.2 Mechanics and Hydrostatics

Stevin wrote *Hydrostatics* in Dutch. (Translated into French by A. Girard, 1634 and into Latin by W. Snell, 1608. His methods influenced Kepler, Cavalieri and others. His contributions to this area rank at a level with Archimedes and he produced several results fundamental to the area. For example, he discovered that the downward pressure of a liquid is independent of the shape of its vessel and depends only on its height and base.

In the *Mechanics* he makes center of gravity calculations. His methods modify and simplify the Archimedian' proof structure. He abandoned the routine of proofs by *reductio ad absurdum*. That is, Stevin accepts that two quantities whose difference can be shown (arbitrarily) small are equal. Here's an example:

**Theorem.** The center of gravity of a triangle lies on its median.



To prove this he inscribes $\Delta ABC$ with quadrilaterals as shown. Divide the median into $2^n$ equal pieces and construct quadrilateral with sides parallel to the base and to the median. Let $\Delta$ denote the area of the triangle and let $I_n$ denote the area of the quadrilaterals, then

$$\Delta - I_n = \Delta/2^n.$$

By Euclid $X - 1$, this can be made arbitrarily small, i.e.

$$\lim_{n\to\infty} \Delta/2^n = 0.$$

Now let $\Delta_1$ and $\Delta_2$ denote the areas of $\Delta ABD$ and $\Delta ADC$. Then

$$\Delta_i - I_n/2 < \Delta/2^n \qquad i = 1, 2.$$

Hence

$$|\Delta_1 - \Delta_2| < \Delta/2^n.$$

(Why?) Thus $\mid \Delta_1 - \Delta_2 \mid$ are arbitrarily small, thus equal. ■

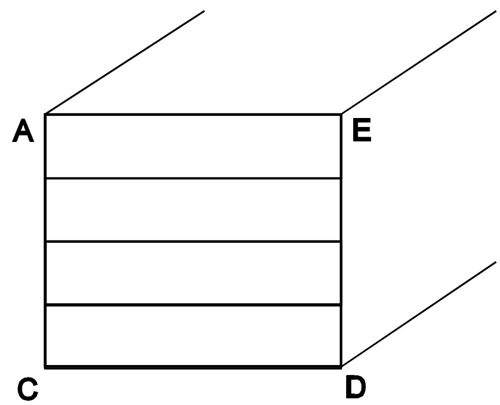
The logic:

1. If two quantities differ they differ by a finite quantity.
2. If these quantities differ by less than any finite quantity, they do not differ.

Yet he doesn't cite this as a general proposition, rather he repeats the argument each time. Another example comes from *Hydrostatics*

(1583). In this work he published the theorem of the triangle of forces. Equivalent to the parallelogram diagram of forces,m this triangle gave a fresh impetus to the study of statics previously founded on the theory of the lever.

**Theorem.** The total pressure on a vertical wall is 1/2 the pressure at the bottom.




Proof. He began his proof by assuming the vertical wall is 1 foot square; area $ACDE = 1$. He divides the wall into horizontal strips of width $1/n \times 1$ feet. He argues that if such a strip were placed horizontally at a depth of $h$ feet below the surface a weight of $(h \times 1/n)ft^3$ of water would rest upon it. Placed vertically the pressure on the strip is

$$h_1 \times \frac{1}{n} < \text{ pressure on strip } < h_2 \times \frac{1}{n},$$

where $h_1 < h_2$ are the depths of the top and bottom of the strip. For the $r^{th}$ strip,

$$h_1 = (r-1)/n \qquad h_2 = r/n.$$

Adding gives

$$\begin{aligned} \frac{1}{n^2}(0+1+\cdots+(n-1)) &< \text{ total pressure (TP) on } ACDE \\ &< \frac{1}{n^2}(1+2+\cdots+n). \end{aligned}$$

Thus,

$$\frac{n(n-1)}{2n^2} < TP < \frac{n(n+1)}{2n^2},$$

and finally

$$\frac{1}{2} - \frac{1}{2n} < TP < \frac{1}{2} + \frac{1}{2n}.$$

## 4 John Napier

John Napier (1550 - 1617) was the son of Sir Archibald Napier was the 7th Laird of Merchiston in Scotland. The family had made its way up over the space of two centuries by service to the King. Sir Archibald, eventually became Master of the Mint.

At the age of 13, Napier entered the St. Salvator's College of University of St. Andrews, though he left without taking a degree having studied for only about one year. Later he went to the continent to study. By 1571 he had returned home for the rest of his days, a scholar competent in Greek. In 1572 he married and after his wife died in 1579, he married again.

Still not many years had past since the Reformation and spirits were high between Protestant and Catholics. Napier, fully caught up in the emotion of the times, spent much of his life immersed in bitter religious dissensions. His antagonism toward the Catholics was legend and he fought passionately in his dealings with the Church of Rome. He sought no quarter and gave none.

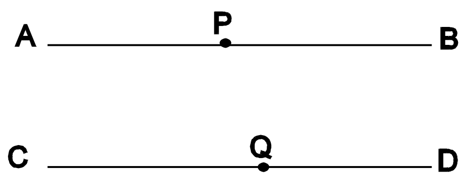
Napier devoted most of his spare time toward the study of mathematics. His particular interest was devising methods of facilitating computation. His greatest achievement toward this end was the creation of **logarithms**. This work, probably begun in 1594, took him fully twenty years to complete.

In *Rabdologiae*, 1617, he included a number of calculating devices, including "Napier's bones," devices that were forerunners of the slide rule. Of course, then, they aided multiplication Book II offers a practical treatment of mensuration rules. Napier was apparently reputed to be a magician, but the evidence is highly doubtful.

In 1614, he wrote *Mirifici Logarithmorum Canonis Descripto* (*Description of the Wonderful Canon of Logarithms*), which contained a brief account showing how to use the tables.

It was in the second book, *Mirifici Logarithmorum Canonis Constructio*, (1619 — note the posthumous date of publication) he explains his idea of using geometry to improve arithmetical computations. The idea follows: Take two lines $AB$ and $CD$

- A variable point $P$ starts at $A$ and moves down $AB$ at a decreasing speed in proportion to the distance from $B$.
- During the same time a point $Q$ moves down $CD$ with a constant speed equal to the starting speed for point $A$.
- Napier called this variable distance $CQ$ the logarithm of $PB$.




In modern terms: Let $x = PB$ and $y = CQ$. Then

$$\frac{dx}{dt} = -x \qquad \frac{dy}{dt} = 10^7,$$

with $x_0 = 10^7$ $y_0 = 0$. So,

$$\frac{dy}{dx} = -\frac{10^7}{x}$$

or

$$y = -10^7 \ln cx$$

when $x_0 = 10^7, y_0 = 0$ so, $c = 10^{-7}$. This gives

$$\begin{aligned} y &= -10^7 \ln(x/10^7) \\ y/10^7 &= \log_{1/e}(x/10^7) \end{aligned}$$

Napier's definition lead directly to logarithms with base $1/e$. Now Napier took effectively $c = 10^7$. This gives: If

$$N = 10^7(1 - 10^{-7})^L$$

then

$L$ is *Napier's logarithm* of $N$.

Now divide $N$ and $L$ by $10^7$

$$\frac{N}{10^7} = \left(1 - \frac{1}{10^7}\right)^{L/10^7} \sim e^{-L}$$

The **product rule** is most important. Suppose

$$N_1 = 10^7(1 - 10^7)^{L_1} \qquad N_2 = 10^7(1 - 10^{-7})^{L_2}.$$

Then

$$N_1 N_2 = 10^7[10^7(1 - 10^{-7})^{L_1+L_2}].$$

Napier's product role has an extra factor of $10^7$.

Napier even coined the word "logarithm" from the Greek: *logos* – ratio and *arithmos* – number.

Napier, of course, was aware of rules for product and quotient. He was unaware of a *base*. The concept of *logarithm function* is implied throughout his work but this notion was not ready for formal definition. Logarithms were a smash hit technology and their use immediately spread throughout Europe, particularly among astronomers whose bane was tedious computation. Even Kepler used them.

Late in life, Napier thought it better to take the $\log 1 = 0$ and $\log 10 = 1$. This he discussed with **Henry Briggs** (1561-1631), a professor of mathematics from Oxford. Briggs returned to Oxford and began the new table. Starting from scratch he computed

$$\sqrt{10} \quad \sqrt{\sqrt{10}}, \quad \sqrt{\sqrt{\sqrt{10}}}, \ldots$$

54 times eventually approximating 1. All calculations were made at 30 decimal places. From these numbers he was able to build a table logarithms of closely spaced numbers.

## 4.1 Jobst Bürgi

In the meantime, actually earlier in 1588, the logarithm idea had occurred to the Swiss mathematician Jobst Bürgi (1552 - 1632). Bürgi published his *Arithmetische und geometrische Progress-Tabulen* in Prague, 1620.

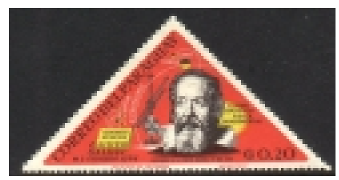
Bürgi used $1 + 10^{-4}$ instead of $1 - 10^{-7}$. He multiplied by $10^8$ instead of $10^7$. If

$$N = 10^8(1 + 10^{-4})^L$$

Bürgi called $10L$ the "red" number corresponding to the "black" number $N$.

Unfortunately for Bürgi, his effort came too late to the attention of the world. In the world of calculation, the work of Napier spread so rapidly that there was little room for another computing technology.

## 5 Galileo Galilei



Galileo Galilei (1564 - 1642), the eldest son of an Italian musician and merchant Vincenzio Galilei, was descended from a Florentine patrician family. His father also made important contributions to music theory and very possible conducted some experiments with Galileo on the tension and pitch of strings. Galileo,himself, was a well known musician and dabbled in business during one part of his career. He enrolled in Pisa in 1581 as a medical student, but left without a degree.

Galileo was soon attracted to mathematics and studied it under Ostillio Ricci in 1583. After he left Pisa, he studied mathematics privately. During 1585-9 Galileo gave lessons in mathematics in Florence and Siena. In 1588, he applied for the chair in mathematics at Bologna but was unsuccessful. However, his "star" was rising and in 1589, he was appointed to the chair in mathematics at Pisa, and in 1592, he was appointed to the chair in mathematics at Padua and remained there until 1610. While in Padua he produced his geometric and military compass and other instruments for sale.

His father, who was not an economic success, died in 1591 leaving him with heavy financial burdens but few assets. He did not marry. However, he had an arrangement with a Venetian woman, Marina Gamba, who bore him two daughters and a son.

Galileo was denounced to the Inquisition in 1615 and that he was tried and condemned by the Inquisition in 1633, living the rest of his life under house arrest. All this was for Copernicanism and for an arrogant attitude toward ecclesiatical authority, not for any heretical[3] theological views.

## 5.1 Galileo's Works

He wrote *De motu* while at Pisa; *Le mechaniche* in the early 90's; work on motion during the first decade of the 17th century, with the composition of a treatise; the *Discorsi* in 1638.

He began his telescopic observations, together with some thought on light and sight beginning in 1609. He wrote *Il saggiatore*, a work of many dimensions, including method and natural on philosophy in general. He also wrote *The Dialogo*, far and away the leading polemic for the Copernican system, in 1632.

In 1610, he was appointed Mathematician and Philosopher to the Grand Duke Cosimo II, with a stipend of 1000 scudi. He was also professor of mathematics at Pisa, without no teaching or residence obligations.

He improved the hydrostatic balance and the proportional compass. He described a crude clock to use with his method of determining longitude. He perfected the crude telescope into an astronomical instrument, and he developed a device, sort of a proto micrometer, to measure diameters of stars and planets. He developed a microscope. He developed a thermoscope.

He developed a circle of young followers mostly in Florence–which included such as Viviani and Torricelli.

Galileo Galilei was largely responsible for reformulating the laws of motion, considered earlier by the Greeks and medieval scholars. He took a geometric approach, not algebraic. Nonetheless he applied mathematics in the service of motion on earth.

Galileo published *Discourses and Mathematical Demonstrations Concerning Two New Sciences*, 1638. Features:

[3] However, Copernicanism could itself be considered heresy and at minimum a challenge to orthodox teachings.

- dialogue form around a Euclidean framework; axiom, etc.
- "Motion is equably or uniformly accelerated which, abandoning rest, adds on to itself equal momenta of swiftness in equal times."
- But this was not his original model of motion. ($\Delta v \propto \Delta x$, not $\Delta v \propto \Delta t$) argues against this by comparing two infinite sets.
- In another place he is concerned with one-one correspondence: integers to squares. He concludes that the attributes of equal to, $<$ or $>$ are problematic for infinite sets.
- He gives a proof of the mean speed rule using an argument just like Oresme's. Like Oresme, he concludes that distance of a body in uniform acceleration travels a distance proportional to the square of the time.
- In all Galileo proves 38 propositions on naturally accelerated motion.
- In the final point he considers motion of a projectile, beginning with the *law of inertia*,
  - 'a body moving on a frictionless plane at a constant velocity will not change its motion'

**Theorem.** When a projectile is carried in motion compounded from equable horizontal and from naturally accelerated downward [motions], it describes a semi parabolic line in its movement.

The discovery of this result came from experiment of rolling balls off a table. This was his previous result about falling bodies ($d = at^2$) plus a knowledge of Apollonius.

He applied this to cannon fire, discovering that the maximum distance is achieved when the firing angle is $45°$.

On factors such as air resistance he says:

- "No firm science can be given of such events of heaviness, speed, and shape which are variable in infinity many ways . . . ."

Two interesting Web sites contain additional information about Galileo. The foremost among them is **The Galileo Project**. Sponsored by Rice University, it is located at http://es.rice.edu/ES/humsoc/Galileo/. The second is the History of Mathematics site at St. Andrews University. This site provides information on almost every mathematician of antiquity up to modern times. It located at http://www-history.mcs.st-andrews.ac.uk/history/. The Galileo page is located at http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Galileo.html. There you will find other links to Galileo sites.

## 6 Renè Descartes (1596-1650)


Renè Descartes (1596-1650) was born into comfortable means. His father was a counselor of the Parlement of Britany– noblesse de la robe. At the age of eight, he entered the Jesuit college of La Flèche in Anjou. There he studied classics, logic, and traditional Aristotelian philosophy. His health was poor and while there he was granted permission to remain in bed until 11 o'clock in the morning. He continued this habit until the year of his death.

In 1617 he set out for the Netherlands and the Dutch army and became a "professional" volunteer who received no pay in the army of the Prince of Orange. He wandered through Europe (at least he travelled in Germany and Italy, in addition to France) during the following eleven years before he settled in the Netherlands in 1628.

Sometime during these military years he inherited one-third of his mother's property, which he sold for about 27,000 livres. He had received enough property from his family that he was free to choose where and how he would live. And he did. In 1628 Descartes left France for the Netherlands in order to isolate himself. It is clear that he lived quite comfortably though he did not aspire to live luxuriously.

From an offer from Queen Christina of Sweden, Descartes moved to Sweden in 1649. (She planned to naturalize him and to incorporate him into the Swedish aristocracy with an estate on conquered German lands.)

In Paris, he was in the circle of Mersenne, Mydorge, Morin, Hardy, Desargues, Villebressieu. In the Netherlands there was first Beeckman and then a network of followers that included Reneri, Regius, Constantijn Huygens, Herreboord, Heydanus, Golius, Schooten, Aemelius. He carried on a mathematical controversy with Fermat.

However, as the grandson and great grandson of physicians, Descartes also gave serious attention to medicine, anatomy, embryology, and meteorology. He spent much time dissecting animals, though nothing new seems to have arisen. He frequently expressed his desire to contribute to the art of curing.

Descartes published his *Discourse on the Method* in 1637 in Leyden. In it he announced his program for philosophical research. In it, he

> " hoped, through systematic doubt, to reach clear and distinct ideas from which it would be possible to deduce innumerably valid conclusions".

He believed everything was explainable in terms of matter and motion. The whole universe was just so. This Cartesian science enjoyed great popularity for almost a century – giving way only to the science of Newton.

The origin of analytic geometry came about while he was seeking to rediscover past truths from the Golden Age. Early on he discovered,

- $v + f = e + 2$ for polyhedra
- construction of roots of a cubic using conics (Manaechmus, Khayyam)

He must have come to analytic geometry sometime around 1628, the time he left France for Holland. There he succeeded in solving the three and four line problems of Pappus (already known). This gave him confidence in his new methods.

*La Geometrie* was one of three appendices to *Discourse*. It is the earliest mathematical text that a present day student can follow without notational problems. Features:

- Uses coordinates to study relations between geometry and algebra. But he seems to use this to further and better make geometric constructions.
- Allows higher plane curves in construction.

- Carefully distinguishes curves that we call **algebraic** (geometric) from others now called **transcendental** (mechanical)
  - geometric – exactly described
  - mechanical – conceived as two separate movements
  - Indeed, his words indicate a sense of self confidence in his understanding of what the Greeks accomplished. The opening lines of Book II reads thusly:

    > “The ancients were familiar with the fact that the problems of geometry may be divided into three classes, namely, plane, solid, and linear problems. This is equivalent ot saying that some problems require only circles and still others require mroe complex curves. I am surprised, however, that they did no go further, and distinguish between different degrees of these more complex curves, nor do I see why they called the latter mechanical, rather than geometrical.
    >
    > If we say that they are called mechanical because some sort of instrument has to be used to describe them, then we must, to be consistent, reject circles and straight lines, since these cannot be sescribed on paper without the use of commasses and ...”

Details of **Book I**:

- solution of quadratics – a geometric construction of the quadratic formula.
- general program –
  - begin with a geometric program
  - reduce it to algebra as far as possible
  - solve and convert back to geometry
- required simplest constructions – for quadratics lines and circles suffice, for cubics and quartics conics are adequate.
- He wrote “Geometry should not include curves that are like strings”, because no rectification can be found.

Here are things absent: rectangular coordinates, formulas for slope, distance, etc., no use of negative abscissas no new curves plotted from coordinates. Concludes Book I with the solution of the three and four line problems of Apollonius (Pappus).

**Book II**: Ovals of Descartes, given two foci $F_1$ and $F_2$

$$mD_1 + nD_2 = k$$

$$m[(x-a)^2 + y^2]^{1/2} + n[(x+a)^2 + y^2]^{1/2} = k$$

**Book III** – was a course on the elementary theory of equations. It included how to

- – discover rational roots, if any
- – deflation
- – determine the number of true $(+)$ and false $(-)$ roots (Descartes *rule of signs*). [The rule: The number $n_p$ of positive zeros of a polynomial $p(x)$ is less than or equal to the number of sign changes $v$ of the coefficients. In any case $v - n_p$ is a non-negative even integer. For example, let $p(x) = x^4 + 3x^2 - x - 5$. The number of sign changes is one; but since $v - n_p$ is even, there must be exactly one positive zero.]
- – find algebraic solutions to cubics and quartics
- – find normals and tangents

Descartes was a professional mathematician, one of our first.

- He is the "father of modern philosophy."
- He presented a changed scientific world view.
- He established a branch of mathematics.

Oresme's latitude of forms more closely anticipates modern mathematical form or function but this played little role in the creation of analytic geometry.

## 7 Pierre de Fermat

Pierre de Fermat (1601-1665) was born into a successful home. His father had a leather business. He was also second consul of Beaumont. His mother brought the social status of the parliamentary noblesse de la robe to the family. He received a solid classical secondary education. After studying with the Franciscans, he then studied with the Jesuits. He may have attended the University of Toulouse. He obtained the degree of Bachelor of Civil Laws from the University of Orleans in 1631. In the Parlement of Toulouse, which was divided according to religion, he was a Catholic counselor. Means of Support: Government, Personal Means He corresponded with Carcavi, Brulart de Saint Martin, Mersenne, Roberval, Pascal, Huygens, Descartes, Frénicle, Gassendi, Lalouvere, Torricelli, Van Schooten, Digby, and Wallis.

Besides mathematical research and his profession Fermat's interests included:

- classical literature
- reconstruction of ancient mathematical texts
- law

Fermat had an enormous ego, and would communicate his theorems without proof and pose many difficult problems. In 1636 he wrote his *Introduction to Plane and Solid Loci*, though it was not published in his lifetime. (At the same time Descartes was preparing his *Discours de la méthode...*)

Because he was familiar with Vièta's work, when he tried to reconstruct Apollonius' work, he replaced Apollonius' geometric analysis with algebraic analysis. This formed the beginning of Fermat's analytic geometry.

### 7.1 àla the Ancients

For example in Fermat's consideration of a theorem on an indeterminate number of points, he is led in the case of just two points to

1. the correspondence between geometric loci and indeterminate algebraic equations in two or more variables, and
2. a system of axes.

In modern terms, we suppose $A = (-a, 0)$ and $B = (0, a)$. Let $x^2+y^2 = r^2$ be the circle (at the bisector of $A$ and $B$), and let $P = (x, y)$ be any point on the circle. Then

$$\begin{aligned} PA^2 + PB^2 &= (x+a)^2 + y^2 + (x-a)^2 + y^2 \\ &= x^2 + 2ax + a^2 + y^2 + x^2 - 2ax + a^2 + y^2 \\ &= 2(x^2 + y^2 + a^2) \\ &= 2(r^2 + a^2) \\ &= 2(AE^2 + EI^2) \end{aligned}$$

In the opening sentence, he says that if in solving algebraically a geometric problem one ends up with an equation in two unknowns, the resulting solution is a locus, straight line or curve. After Vièta, Fermat sketched the simplest case of a linear equation: in Latin

$$D \text{ in } A \text{ aequetur } B \text{ in } E$$

(or $Dx = By$ in modern notation).

He then shows a new method to prove the

**Theorem:** Given any number of fixed lines, in a plane, the locus of a point such that the sum of any multiples of the segments drawn at given angles from the point to the given lines is constant, is a straight line.

This follows from the fact that the segments are linear functions of the coordinates and Fermat's proposition that first degree equations represent straight lines. Fermat then shows that $xy = k^2$ is a **hyperbola** and the form $xy + a^2 = bx + cy$ can be reduced to $xy = k^2$ by a change of axes. He establishes similar results for the parabola and for the ellipse. His "crowning" result is this

**Theorem.** Given any number of fixed lines, the locus of a point such that the sum of the squares of the segments drawn at given right angles from the point to the lines is constant, is a solid locus (ellipse).

This would be nearly impossible to prove without analytic geometry. We have, in modern terms

$$\sum_{n=1}^{N}\left[\frac{a_n x + b_n y + c_n}{\sqrt{a_n^2 + b_n^2}}\right]^2 = k$$

This gives an ellipse.

In Summary, Fermat's exposition and clarity was better than Descartes'. His analytic geometry is closer to our own. He uses rectangular coordinates.

## 7.2 Fermat's Theory of Numbers.

Fermat's probability theory and analytic geometry were virtually ignored in his lifetime, and for almost 100 years after. He was given to **secrecy** of methods, and proofs are scarce. His theory of numbers was widely circulated by Mersenne and others. It began with a study of **perfect** numbers. He proved three propositions on this – communicated to Mersenne in 1640. The first result was

(1) Theorem. If $n$ is not prime $2^n - 1$ is not prime.

Proof.

$$\begin{aligned} 2^n - 1 &= 2^{rs} - 1 = (2^r)^s - 1 \\ &= (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} \cdots + 2^r + 1). \end{aligned}$$

Thus the basic question is reduced to finding prime $p$ for which $2^p - 1$ is prime. (These are called Mersenne primes.) His next two theorems were

(2) Theorem. If $p$ is odd prime, then $2p$ divides $2^p - 2$ or $p$ divides $2^{p-1} - 1$.

(3) Theorem. The only possible divisors of $2^p - 1$ have the form $2pk + 1$.

All given without proofs! He gave a few examples. Of course, Theorem 3 reduces the number of divisors one must check for primality of

$2^p - 1$. There is a more general result he communicated to **Bernard Frenicle de Bessy** (1612-1676)

Theorem (Fermat's Little Theorem.) If $p$ is any prime and $a$ any positive integer then $p$ divides $a^p - a$.

Thus,

$$a^p = a \bmod p.$$

If $a$ and $p$ are relatively prime, then

$$a^{p-1} = 1(\mathrm{mod} p).$$

The original form of Fermat's little theorem is that given a prime $p$ and integer $a$, it then $a^{p-1} = 1\,(\mathrm{mod}\, m)$. Euler gave a generalization of this result as follows.

Theorem. (Fermat's Little Theorem) Suppose $a$ and $m$ are integers and $\phi\,(m)$ is the Euler totient function. If $(a, m) = 1$ then

$$a^{\phi(m)} = 1 \ (\mathrm{mod}\, m)$$

Proof. Let $r_1,\, r_2,\, \ldots\,,\, r_k$ be the integers $< m$ that are relatively prime to $m$. Now define $r_j a = s_j m + k_j$. We know that $k_j < m$ for $j = 1,\, 2,\, \ldots\,, m$, and we claim that each of the $k_j$ are distinct. For suppose without loss of generality that $k_i = k_j$. Then $(r_i - r_j)\, a = (s_i - s_j)\, m$. Since $(a, m) = 1$ we must have that $m$ divides $(r_i - r_j)$, which is impossible. We also conclude that $(k_j,\, m) = 1$, for each $j = 1,\, 2,\, \ldots\,, m$. For if not, there is an index $i$ and a divisor of both $k_i$ and $m$ which also divides $r_i$. This is impossible. We conclude that the sets $\{r_j\}_{j=0}^k = \{k_j\}_{j=0}^k$. Next

$$\begin{aligned}
\Pi_{j=0}^k\,(r_j a) &= \Pi_{j=0}^k\,(r_j)\,\Pi_{j=0}^k a = \Pi_{j=0}^k\,(r_j)\,a^{\phi(m)} \\
&= \Pi_{j=0}^k\,(s_j m + k_j) \\
&= mb + \Pi_{j=0}^k\,(k_j) = mb + \Pi_{j=0}^k\,(r_j) \\
&= \Pi_{j=0}^k\,(r_j)\,(\mathrm{mod}\ m)
\end{aligned}$$

Now since $\left(\Pi_{j=0}^k\,(r_j)\,,\, m\right) = 1$, it follows that

$$a^{\phi(m)} = 1\,(\mathrm{mod}\ m)$$

Fermat gave no clue to discovery. However, this "little theorem" turned out to be very useful in number theory yielding many, many applications. The first published proof, given by Euler (1732), was the more general form. (Leibnitz, however, left an earlier proof in manuscript form.)

Fermat also said that he believed all numbers of the form

$$2^{2^n} + 1$$

were prime and announced a proof in 1659. (In 1732, Euler discovered that $2^{2^5}+1$ is composite.) It is likely Fermat used his method of "infinite descent" using $n = 0, 1, 2, 3, 4$ to arrive at the general conclusion.

Incidentally, using this method, which historically has been very important, Fermat showed that there is no integral right triangle whose area is a square.

$$x^2 + y^2 = z^2 \qquad \text{(sides)}$$
$$\frac{1}{2}xy = w^2 \qquad \text{(area)}$$

Finally we note **Fermat's Last Theorem**:

$$x^n + y^n = z^n$$

has no integral solutions if $n \geq 3$. It was resolved in the negative by Andrew Wiles in 1995 — as a part of a larger mathematical program. After Fermat, number theory faded only to re-emerge a century later.

## 8 Andriaan van Roomen

Andriaan van Roomen (1561-1615), a Belgian physician and mathematician occupies a minor but interesting place in mathematical history. He studied in Louvain and at the Jesuit College in Cologne. In Würzburg, where he was a professor of medicine, he published a continuing series of medical theses defended by his students. A prolific author, Roomen wrote also on astronomy and natural philosophy. As with medicine, his opinions in these fields were traditional. As a mathematician he was especially concerned with trigonometry. He calculated the sides of the regular polygons, and from the polygon with 216 sides calculated the value of pi to sixteen places, achieving the same accuracy as Al Kashi. He also wrote a commentary on algebra. He corresponded with a considerable number of the mathematicians and scientists of his day, including Ludolph van Ceulen[4] and Viete. He also posed a problem that was solved by Vieta, as mentioned earlier.

## 9 Remark.

The author is indebted to Jeff Miller, a mathematics teacher at Gulf High School in New Port Richey, Florida, whose collection of images of significant mathematicians is located at http://members.tripod.com/jeff560/index.html has provided a number of the images in the chapters of this online course. The author is also indebted to Maiken Naylor who maintains a large collection of science related stamps at the University of Buffalo. (See, http://ublib.buffalo.edu/libraries/units/sel/exhibits/stamps/)

[4]Ludolph van Ceulen (1540 - 1610) computed $\pi$ to 35 places.