Egyptian Mathematics

Our first knowledge of mankind's use of mathematics beyond mere counting comes from the Egyptians and Babylonians. Both civilizations developed mathematics that was similar in some ways but different in others. The mathematics of Egypt, at least what is known from the papyri, can essentially be called applied arithmetic. It was practical information communicated via example on how to solve specific problems.

Pyramids at Giza

This point, that mathematics was communicated by example, rather than by principle, is significant and is different than today's mathematics that is communicated essentially by principle with examples to illustrate principles. The reasons for this are unknown but could be due partly to the fact that symbolism, the medium of principles, did not exist in these early times. Indeed, much of mathematics for many centuries was communicated in this way. It may be much easier to explain to a young student an algorithm to solve a problem and for them to learn to solve like problems, than to explain the abstract concept first and basing examples upon this concept.

1 Basic facts about ancient Egypt.

Egyptian hieroglyphics are in great abundance throughout Egypt. They were essentially indecipherable until 1799 when in Alexandria the trilingual **Rosetta Stone** was discovered. The Rosetta stone, an irregularly shaped tablet of black basalt measuring about 3 feet

9 inches by 2 feet 4 inches, was found near the town of Rosetta (Rashid) just a few miles northwest of Alexandria. Written in the two languages (Greek and Egyptian but three writing systems (hieroglyphics, its cursive form demotic script, and Greek, it provided the key toward the deciphering of hieroglyphic writing. The inscriptions on it were the benefactions conferred by Ptolemy V Epiphanes (205 - 180 BCE) were written by the priests of Memphis. The translation was primarily due to Thomas Young[1] (1773 - 1829) and

Temple at Al Karnak

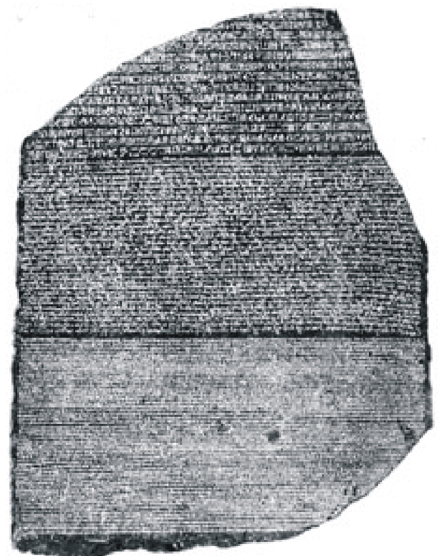
Jean Francois Champollion (1790-1832)[2] (1790-1832), who, very early in his life was inspired to Egyptology by the French mathematician Jean Baptiste Joseph Fourier (1768 - 1830). Champollion completed the work begun by Young and correctly deciphered the complete stone. An Egyptologist of the first rank, he was the first to recognize the signs could be alphabetic, syllabic, or determinative (i.e. standing for complete ideas) He also established the original language of the Rosetta stone was Greek, and that the hieroglyphic text was a translation from the Greek. An unusual aspect of hieroglyphics is that they can be

[1]English physician and physicist established the principle of interference of light and thus resurrected the century-old wave theory of light.

[2]French historian and linguist who founded scientific Egyptology. Academically prodigious, he had already mastered six ancient Oriental languages by the age of 16. At 19, he was appointed professor of history at the lycé of Grenoble, where he was to remain for eight years. Deciphering hieroglyphics became his constant preoccupation

read from left to right, or right to left, or vertically (top to bottom). It is the orientation of the glyphs that gives the clue; the direction of people and animals face toward the beginning of the line.

For the Egyptians writing was an esthetic experience, and they viewed their writing signs as "God's words." This could explain the unnecessary complexity, in face of the fact that obviously simplifications would certainly have occurred if writing were designed for all citizens.



Rosetta Stone

The demotic script was for more general use, the hieroglyphics continued to be used for priestly and formal applications.

The Egyptians established an annual calendar of 12 months of 30 days each plus five feast days. Religion was a central feature of Egyptian society. There was a preoccupation with death. Many of Egypt's greatest monuments were tombs constructed at great expense, and which required detailed logistical calculations and at least basic geometry.

Construction projects on a massive scale were routinely carried out. The logistics of construction require all sorts of mathematics. You will see several mensuration (measurement) problems, simple algebra problems, and the methods for computation.

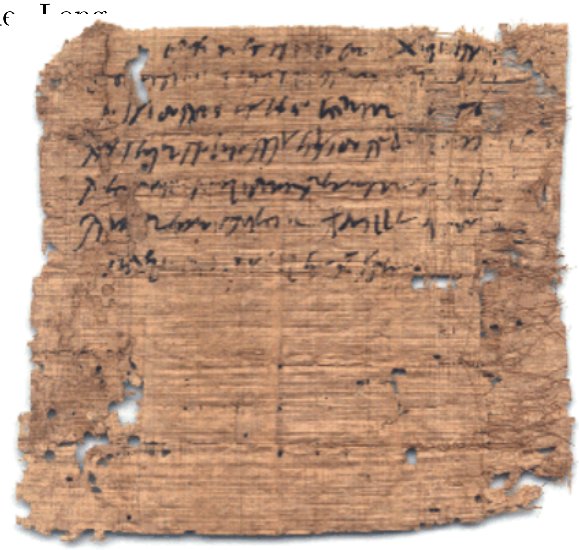
Our sources of Egyptian mathematics are scarce. Indeed, much of our knowledge of ancient Egyptian mathematics comes not from the hieroglyphics[3] (carved sacred letters or sacred letters) inscribed on the hundreds of temples but from two papyri containing collections

[3]The words "hieroglyph" or "hieroglyphic" are derived from the Greek words *grammata hiera* or *grammata hieroglyphika* respectively.

of mathematical problems with their solutions.

- The **Rhind Mathematical Papyrus** named for A.H. Rhind (1833-1863) who purchased it at Luxor in 1858. Origin: 1650 BCE but it was written very much earlier. It is 18 feet long and 13 inches wide. It is also called the **Ahmes Papyrus** after the scribe that last copied it.
- The **Moscow Mathematical Papyrus** purchased by V. S. Golenishchev (d. 1947). Origin: 1700 BC. It is 15 ft long and 3 inches wide. Two sections of this chapter offer highlights from these papyri.

Papyrus, the writing material of ancient times, takes its name from the plant from which it is made. Long cultivated in the Nile delta region in Egypt, the Cyperus papyrus was grown for its stalk, whose inner pith was cut into thin strips and laid at right angles on top of each other. When pasted and pressed together, the result was smooth, thin, cream-colored papery sheets, normally about five to six inches wide. To write on it brushes or styli, reeds with crushed tips, were dipped into ink or colored liquid.



From the Duke Papyrus Archive*

A remarkable number of papyri, some dating from 2,500 BCE, have been found, protected from decomposition by the dry heat of the region though they often lay unprotected in desert sands or burial tombs.

* See the URL: http://odyssey.lib.duke.edu/papyrus/texts/homepage.html

2 Counting and Arithmetic — basics

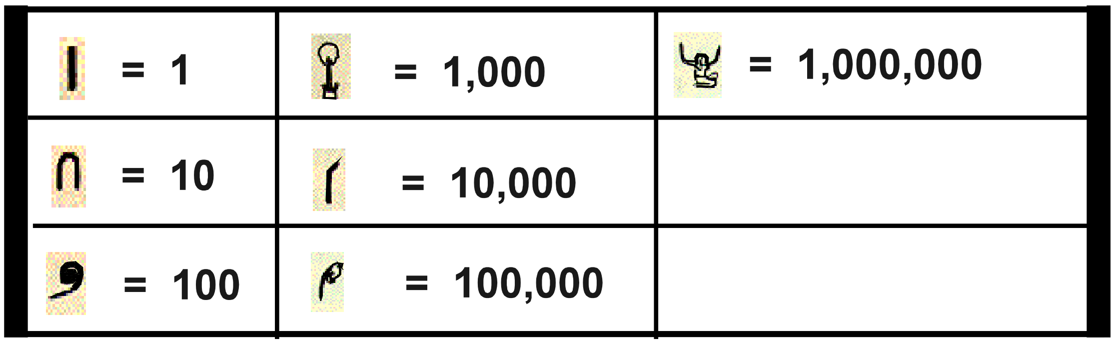
The Egyptian counting system was decimal. Though non positional, it could deal with numbers of great scale. Yet, there is no apparent way to construct numbers arbitrarily large. (Compare that with modern systems, which is positional, which by its nature allows and economy for expressing huge numbers.)

The number system was decimal with special symbols for 1, 10, 100, 1,000, 10,000, 100,000, and 1,000,000. Addition was accomplished by grouping and regrouping. Multiplication and division were essentially based on binary multiples. Fractions were ubiquitous but only unit fractions, with two exceptions, were allowed. All other fractions were required to be written as a sum of unit fractions. Geometry was limited to areas, volumes, and similarity. Curiously, though, volume measures for the fractional portions of the *hekat* a volume measuring about 4.8 liters, were symbolically expressed differently from others.

Simple algebraic equations were solvable, even systems of equations in two dimensions could be solved.

Symbolic notation for numbers.

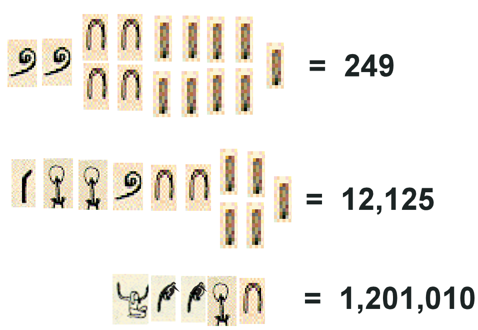
Egyptian Hieroglyphic Numbers



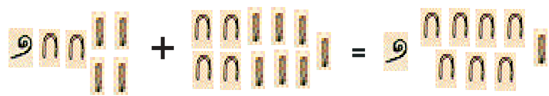
1	=	vertical stroke
10	=	heal bone
100	=	a snare
1,000	=	lotus flower
10,000	=	a bent finger
100,000	=	a burbot fish
1,000,000	=	a kneeling figure

Note though that there are numerous interpretations of what these hieroglyphs might represent.

Numbers are formed by grouping.



Addition is formed by grouping



124 + 47 = 171

Note alternate forms for these numbers.

Multiplication is basically binary.

Example: Multiply: 47×24

47	$\times$	24	
47		1	doubling process
94		2	
188		4	
376		8	*
752		16	*

Selecting 8 and 16 (i.e. $8 + 16 = 24$), we have

$$\begin{aligned} 24 &= 16 + 8 \\ 47 \times 24 &= 47 \times (16 + 8) \\ &= 752 + 376 \\ &= 1128 \end{aligned}$$

Division is also basically binary.

Example: Divide: $329 \div 12$

329	÷	12		
12		1	doubling	329
24		2		-192
48		4		137
96		8		-96
192		16		41
384		32		-24
				17
				-12
				5

Now

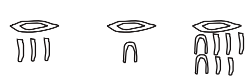
$$\begin{aligned} 329 &= 16 \times 12 + 8 \times 12 + 2 \times 12 + 1 \times 12 + 5 \\ &= (16 + 8 + 2 + 1) \times 12 + 5 \end{aligned}$$

So,

$$329 \div 12 = 27\,\frac{5}{12} = 27 + \frac{1}{3} + \frac{1}{12}.$$

Obviously, the distributive laws for multiplication and division were well understood.

Fractions It seems that the Egyptians allowed only unit fractions, with just two exceptions, $\frac{2}{3}$ and $\frac{3}{4}$. All other fractions must be converted to unit fractions. The symbol for unit fractions was a flattened oval above the denominator. In fact, this oval was the sign used by the Egyptians for the "mouth." In the case of the volume measure ***hekat***, the commonly used fractional parts of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$,and $\frac{1}{64}$, were denoted by parts of the symbol for the *Horus-eye*, symbolized as .[4] For ordinary fractions, we have the following.



1/3 1/10 1/25

There were special symbols for the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, of which one each of the forms is shown below.

[4]See Ifrah, p 169.

1/2 2/3 3/4

All other fractions must be converted to unit fractions. For example:

$$\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$$

Obviously there are but two cases to consider, unit fractions and fractions with numerator two. All fractions can be reduced to a sum of such fractions. Ahmes gives a table of unit fractions decompositions for fractions with numerator two.

$\frac{2}{n}$	$1/p$	$+$	$1/q$	$+$	$1/r+\ldots$
5	3		15		
7	4		28		
9	6		18		
11	6		66		
13	8		52		104
15	10		30		
$\vdots$					

- Decompositions are not necessarily unique. The Egyptians did *favor* certain fractions and attempt to use them when possible. For example, they seems to prefer taking halves when possible. Thus the representation for $2/15$ as

$$\frac{2}{15} = 1/30 + 1/10$$

- The exact algorithm for determination for the decomposition is unknown and this is an active topic of research today. However, in other papyri, there is some indication of the application of the formula

$$\frac{2}{p \cdot q} = \frac{1}{p \cdot \frac{p+q}{2}} + \frac{1}{q \cdot \frac{p+q}{2}}$$

being used. It gives some, but not all, of the table, and certainly does not give decompositions into three or more fractions.

- It seems certain that the Egyptians understood general rules for handling fractions.

3 The Ahmes Papyrus

The Ahmes was written in hieratic, and probably originated from the Middle Kingdom: 2000-1800 BC. It claims to be a "thorough study of all things, insight into all that exists, knowledge of all obscure secrets." In fact, it is somewhat less. It is a collection of exercises, substantially rhetorical in form, designed primarily for students of mathematics. Included are exercises in

- fractions
- notation
- arithmetic
- algebra
- geometry
- mensuration

The practical mathematical tools for construction?

To illustrate the level and scope of Egyptian mathematics of this period, we select several of the problems and their solutions as found in the two papyri. For example, beer and bread problems are common in the Ahmes.

Problem 72. How many loaves of "strength" 45 are equivalent to 100 loaves of strength 10? Fact:

$$\text{strength} := \frac{1}{\text{grain density}}$$

Invoking the rule of three[5], which was well known in the ancient world, we must solve the problem:

$$\frac{x}{45} = \frac{100}{10}$$

Answer: $x = 100/10 \times 45 = 450$ loaves.

Problem 63. 700 loaves are to be divided among recipients where the amounts they are to receive are in the continued proportion

$$\frac{2}{3} : \frac{1}{2} : \frac{1}{3} : \frac{1}{4}$$

Solution. Add

$$\frac{2}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{7}{4}.$$

$$\begin{aligned}
\frac{700}{7/4} &= 700 \cdot \frac{4}{7} \\
&= 700(\frac{2}{7} + \frac{2}{7}) \\
&= 700(\frac{2}{7} + \frac{1}{7} + \frac{2}{28} + \frac{1}{14}) \\
&= 700(\frac{1}{2} + \frac{1}{14}) \\
&= 350 + 50 \\
&= 400
\end{aligned}$$

The first value is 400. This is the base number. Now multiply each fraction by 400 to obtain the recipient's amount. Note the algorithm nature of this solution. It reveals no principles at all. Only when converting to modern notation and using modern symbols do we see that this is correct We have

$$\frac{x_1}{x_2} = \frac{\frac{2}{3}}{\frac{1}{2}}, \quad \frac{x_2}{x_3} = \frac{\frac{1}{2}}{\frac{1}{3}},$$

[5]The rule of three was the rule to determine the fourth and unknown quantity in the expression $\frac{a}{b} = \frac{c}{d}$ in which the other three are known

etc. This will be the case if there is a base number a such that

$$x_1 = \frac{2}{3}a$$
$$x_2 = \frac{1}{2}a$$
$$x_3 = \frac{1}{3}a$$
$$x_4 = \frac{1}{4}a$$

Thus

$$x_1 + x_2 + x_3 + x_4 = (\frac{2}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})a = 700$$

Now add the fractions to get $\frac{7}{4}$ and solve to get

$$a = 400.$$

Now compute $x_1,\ x_2,\ x_3,\ x_4$. This problem seems to indicate a type of hierarchical chain for the distribution of goods was relatively common. Similar problems are relatively rare in modern texts.

The solution of **linear algebra** problems is present in the Ahmes. Equations of the modern form

$$x + ax = b \quad \text{or} \quad x + ax + bx = x,$$

where a, b, and c are known are solved. The unknown, x, is called the **heep**. Note the rhetorical problem statement.

Problem 24. Find the heep if the heap and a seventh of the heep is 19. (Solve $x + x/7 = 19$.)

Method. Use the method of **false position**. Let g be the guess. Substitute $g + ag = c$. Now solve $c \cdot y = b$. Answer: $x = g \cdot y$. **Why?**

Solution. Guess $g = 7$.

$$7 + 1/7 \cdot 7 = 8$$

$$19 \div 8 = 2 + \frac{3}{8} = 2 + \frac{1}{4} + \frac{1}{8}$$

Answer:

$$7 \cdot (2 + \frac{3}{8}) = 7(2 + \frac{1}{4} + \frac{1}{8}) = 16 + \frac{1}{2} + \frac{1}{8}$$

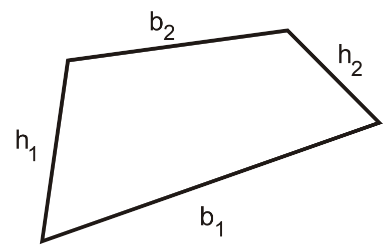
Geometry and Mensuration Most geometry is related to mensuration. The Ahmes contains problems for the areas of

- isosceles triangles (correct)
- isosceles trapezoids (correct)
- quadrilaterals (incorrect)
- frustum (correct)
- circle (incorrect)
- curvilinear areas

In one problem the area for the quadrilateral was given by

$$A = (\frac{b_1 + b_2}{2})(\frac{h_1 + h_2}{2})$$

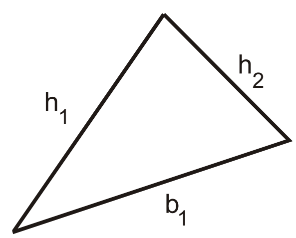
which of course is wrong in general but correct for rectangles. Yet the "Rope stretchers" of ancient Egypt, that is the land surveyors, often had to deal with irregular quadrilaterals when measuring areas of land. This formula is quite accurate if the quadrilateral in question is nearly a rectangle.



Quadrilateral

The area for the triangle was given by replacement $b_2 = 0$ in the quadrilateral formula

$$A = (\frac{b_1}{2})(\frac{h_1 + h_2}{2})$$



Triangle

On Rigor. There is in Egyptian mathematics a search for relationships, but the Egyptians had only a vague distinction between the **exact** and the **approximate**. Formulas were not evident. Only solutions to specific problems were given, from which the student was left to generalize to other circumstances. Yet, as we shall see, several of the great Greek mathematicians, Pythagoras , Thales, and Eudoxus to name three, studied in Egypt. There must have been more there than student exercises to learn!

Problem 79. This problem cites only "seven houses, 49 cats, 343 mice, 2401 ears of spelt, 16,807 hekats."

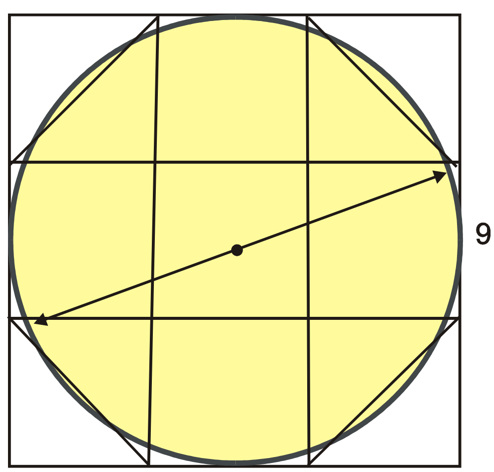
Note the similarity to our familiar nursery rhyme:

As I was going to St. Ives,
I met a man with seven wives;
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kits.
Kits, cats, sacks, and wives,
How many were going to St. Ives?

This rhyme asked for the very impractical sum of all and thus illustrates some knowledge and application of geometric progressions.

Problem 50. A circular field of diameter 9 has the same area as a square of side 8. This gives an effective $\pi = 3\,\frac{1}{6}$.

Problem 48 gives a hint of how this formula is constructed.



Side length = 9

Trisect each side. Remove the corner triangles. The resulting octagonal figure approximates the circle. The area of the octagonal figure is:

$$9 \times 9 - 4(\frac{1}{2} \cdot 3 \cdot 3) = 63 \approx 64 = 8^2$$

Thus the number

$$4(\frac{8}{9})^2 = 3\,\frac{13}{81}$$

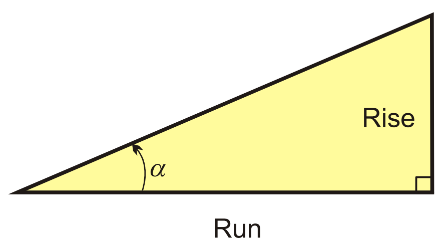
plays the role of π. That this octagonal figure, whose area is easily calculated, so accurately approximates the area of the circle is just plain good luck. Obtaining a better approximation to the area using finer divisions of a square and a similar argument is not simple.

Geometry and Mensuration

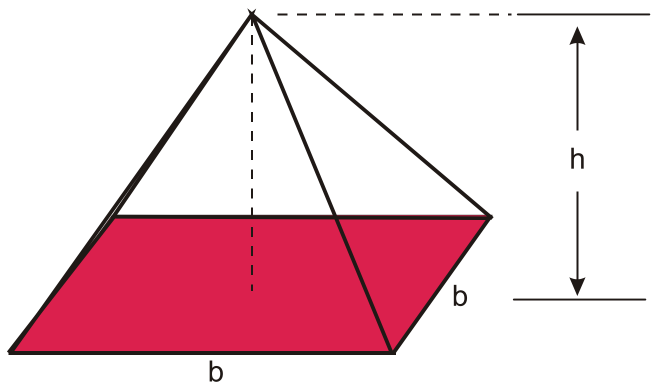
Problem 56 indicates an understanding of the idea of geometric similarity. This problem discusses the ratio

$$\frac{\text{rise}}{\text{run}}$$

The problem essentially asks to compute the $\cot\alpha$ for some angle α. Such a formula would be need for building pyramids.



Note the obvious application to the construction of a pyramid for which the formula for the volume, $V = \frac{1}{3}b^2h$, was known. (How did they find that?)



Pyramid

Geometry and Mensuration

The are numerous myths about the presumed geometric relationship among the dimensions of the Great Pyramid. Here's one:

[perimeter of base]=
[circumference of a circle of radius=height]

Such a formula would yield an effective $\pi = 3\frac{1}{7}$, not $\pi = 3\frac{1}{6}$, as already discussed.

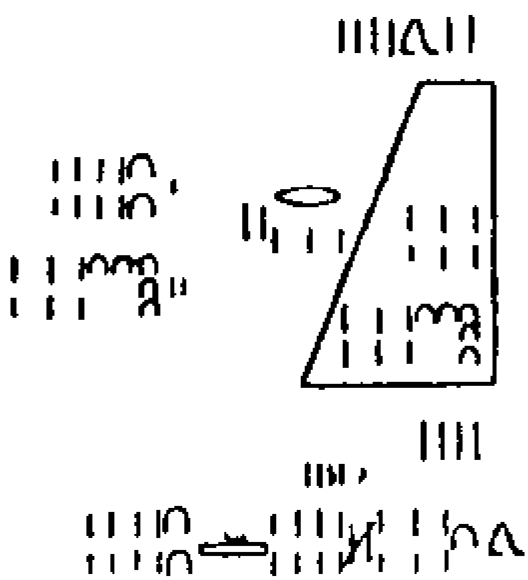
4 The Moscow Papyrus

The Moscow papyrus contains only about 25, mostly practical, examples. The author is unknown. It was purchased by V. S. Golenishchev (d. 1947) and sold to the Moscow Museum of Fine Art. Origin: 1700 BC. It is 15 feet long and about 3 inches wide.

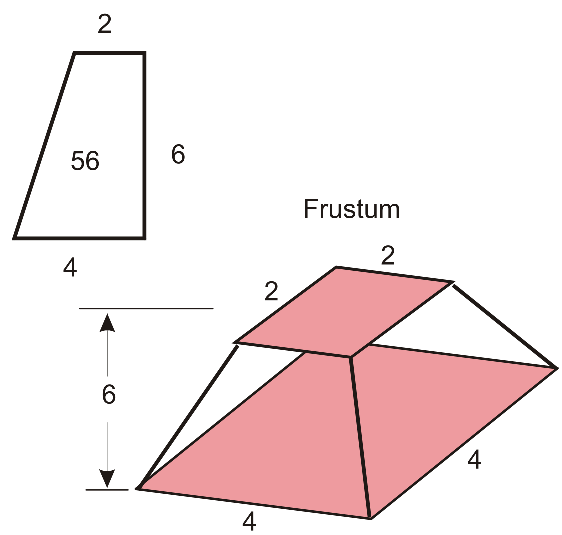
Problem 14. Volume of a frustum. The scribe directs one to square the numbers two and four and to add to the sum of these squares the product of two and four. Multiply this by one third of six. “See, it is 56; your have found it correctly.” What the student has been directed to compute is the number

$$V = \frac{1}{3} \cdot 6(2^2 + 4^2 + 2 \cdot 4) = 56$$

Here’s the picture that is found in the Moscow Papyrus.



Here's the modern version of the picture and a perspective drawing.



The general formula for a frustum was evidently known to the Egyptians. It is:

$$V = \frac{1}{3}h(b_1^2 + b_1 b_2 + b_2^2)$$

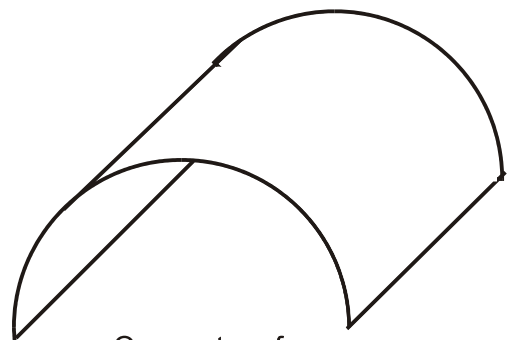
Taking $b_1 = 0$, we get the formula

$$V = \frac{1}{3}hb^2$$

This was evidently known also.

Question. Speculate on how the Egyptians could have known the formula for a frustum, given that its derivation depends on the methods of modern calculus.

Problem 10 Compute the surface area of a Quonset type hut roof, which is the earliest estimation of curvilinear area.



Quonset roof

5 Summary of Egyptian Mathematics

In the few bullet items below we give a summary of known Egyptian mathematical achievements. Records of conquests of pharohs and other facts of Egyptian life are in abundance throughout Egypt, but of her mathematics only traces have been found. These fragments, from a civilization that lasted a millennium longer than the entire Christian era, that undertook constructions projects on a seen not seen again[6] until this century, and that created abundance from a desert, allow only the following conclusions.

- Egyptian mathematics remained remarkably uniform throughout time.
- It was built around addition.
- Little theoretical contributions were evident. Only the slightest of abstraction is evident. Yet exact versions of difficult to find formulas were available.
- It was substantially practical. The texts were for students. No "principles" are evident, neither are there laws, theorems, axioms postulates or demonstrations; the problems of the papyri are examples from which the student would generalize to the

[6]excluding the Great Wall of China constructed in third century BCE

actual problem at hand. The papyri were probably not written for self-study. No doubt there was a teacher present to assist the student learning the examples and then giving "exercises" for the student to solve.

- There seems to be no clear differentiation between the concepts of exactness and approximate.
- Elementary congruencies were used only for mensuration.

Yet, there must have been much more to Egyptian mathematics. We know that Thales, Pythagoras and others visited Egypt to study. If there were only applied arithmetic methods as we have seen in the papyri, the trip would have had little value. But where are the records of achievement? Very likely, the mathematics extant was absorbed into the body of Greek mathematics — in an age where new and better works completely displaced the old, and in this case the old works written in hieroglyphics. Additionally, the Alexandrian library, one place where ancient Egyptian mathematical works may have been preserved, was destroyed by about 400 CE.