Written Final: MAT 229, Spring 2025

Name:

Show your work to receive credit; cross out (don't erase). You may not skip problem 1, but are to select 7 of the remaining 10 problems to do (skip 3). Good luck!

Draw your graphs on this sheet. Please carefully separate problems on your answer sheets so that they're easier to grade. (Remember to keep the grader happy!)

1. (20 pts) Consider the function $f(x)=x\ln(x)$. FYI, $\ln(2)\approx .7$, $\ln(3)\approx 1.1$.

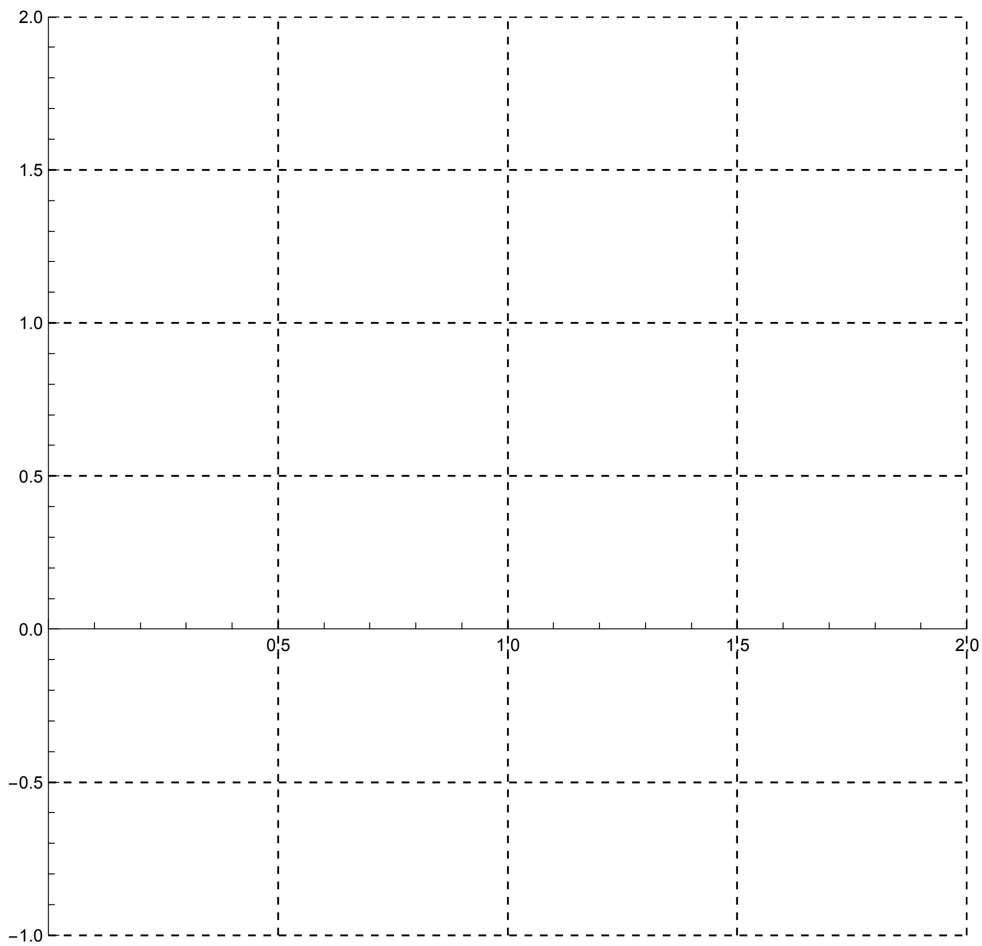
1.1. Compute the limit: $\lim_{x\to 0^+} f(x)$

1.2. Compute the derivative of $f(x)$, and find the value of x at which it equals zero (you shouldn't need a calculator).

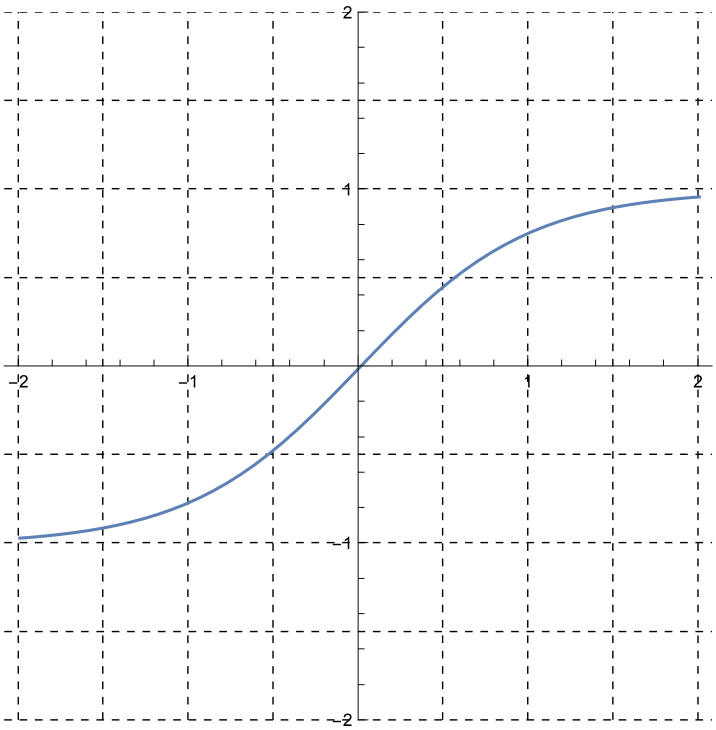
1.3. Compute the integral using an appropriate technique of integration: $\int f(x)\,dx$

1.4. Compute the integral numerically, using trapezoidal and two subintervals: $\int_1^2 f(x)\,dx$. Get the answer into a format that one could compute easily with a calculator.

1.5. Sketch the function and its derivative on the axes below, to the right. A rough sketch is okay, but it should incorporate what you know or have learned above.



2. (10 pts) Function *f(x)* is invertible, odd, monotonic, and its graph is below:



2.1. If f has horizontal asymptotes of ±1, describe the domain and range of its inverse f^{-1}.

2.2. **Carefully** add the graph of its inverse, f^{-1}, to the plot above.

2.3. Given that $f(\frac{3}{2})=\frac{9}{10}$, and that $f'(\frac{3}{2})=\frac{1}{8}$; write the equations of the tangents line to f at $x=\frac{3}{2}$ and to f^{-1} at $x=\frac{9}{10}$, in point-slope form.

2.4. **Carefully** draw both tangent lines on the axes above.

3. (10 pts) Compute the integral using appropriate techniques of integration.

3.1. $\int x\, e^{-x}\, dx$

3.2. $\int \frac{1}{\sqrt{4-x^2}}\, dx$

4. (10 pts) Determine, by any of our methods, whether the following series converge or not, and, if they converge, whether they do so absolutely or conditionally:

4.1. $\sum_{k=1}^{\infty} \frac{\sin(k)}{1+k^2}$

4.2. $\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k+1}$

5. (10 pts) We learned that the Maclaurin series for $\cos(x)$ is $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$.

5.1. Determine its interval of convergence using a ratio test.

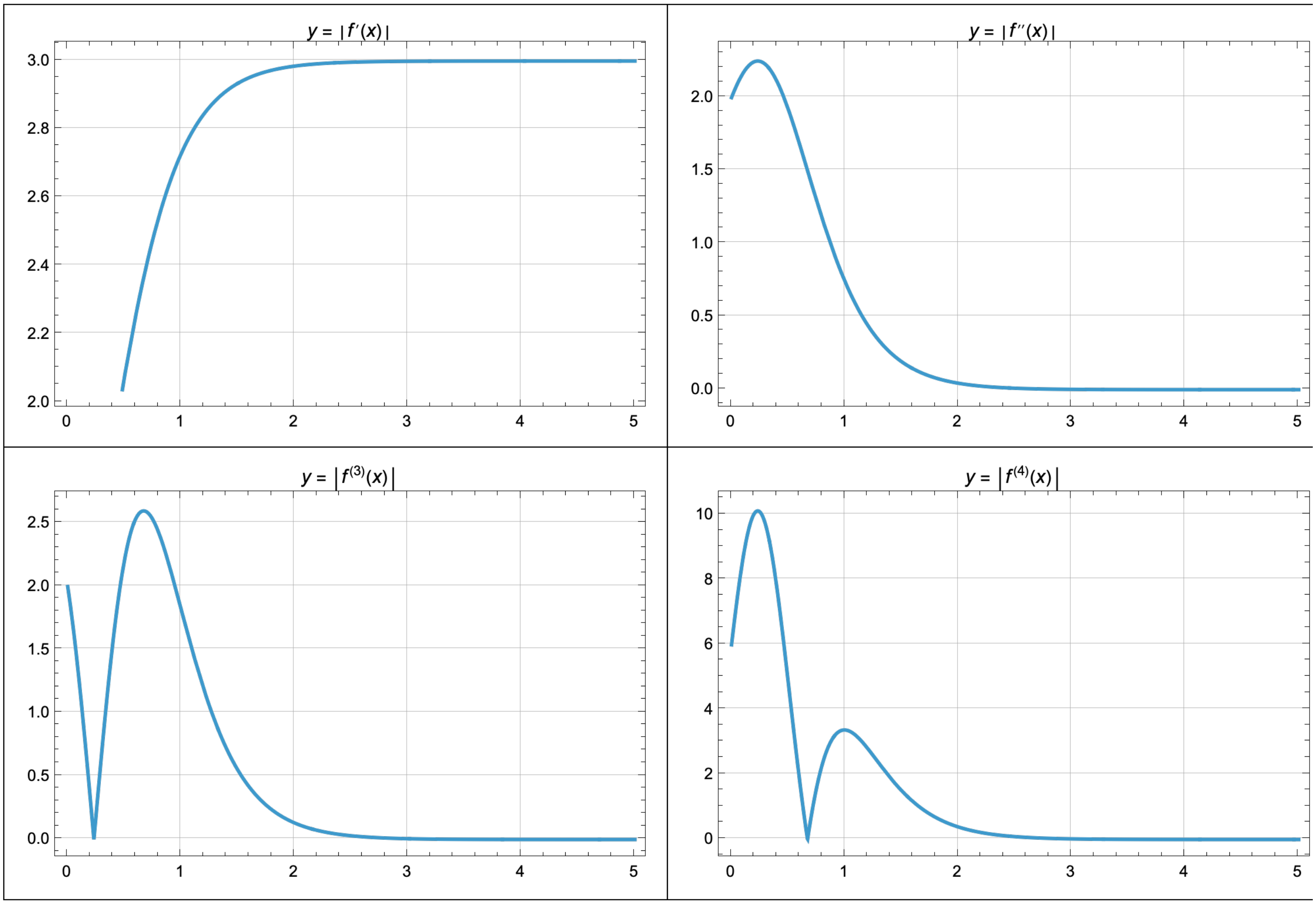
5.2. Use the Taylor remainder theorem to bound the error we make in estimating $\cos(\frac{\pi}{4})$ with the corresponding 8^{th} degree Taylor polynomial. You do not actually need to compute the value of the bounding error -- but you must explain how you bound the appropriate derivative, and put everything together -- everything up to where you'd need a calculator.

5.3. Write the Maclaurin series for $\cos(\sqrt{x})$.

6. (10 pts) Consider the definite integral

$$\int_0^5 \ln\left(2+e^{3x}\right) dx.$$

The graphs of the various derivatives for the integrand $f(x)=\ln\left(2+e^{3x}\right)$ are given below.



6.1. Determine a formula for n for which the following methods would estimate the integral with absolute error less than 0.001. Get the result as far as you can without a calculator.

6.1.1. midpoint method

6.1.2. Simpson's rule

6.2. Suppose we'd made a mistake, and the integral was actually $\int_0^5 \ln\left(2*e^{3x}\right) dx$. Compute the integral analytically.

7. (10 pts) Determine, by any of our methods, the smallest value of n such that the error in computing an approximation to the infinite series S with the partial sum S_n will be within 0.0001 of the correct value. If you get the problem down to, e.g. $n > \ln(2000)$, you're done: see how far you can get without resorting to a calculator.

7.1. $S_n = \sum_{k=1}^{n} k\,e^{-k}$.

7.2. $S_n = \sum_{k=1}^{n} \frac{1}{2^{k+3}}$

8. (10 pts) Consider the parametric equations, given by $x(t) = \frac{1}{2}t^2$ and $y(t) = \frac{1}{3}t^3$.

8.1. Eliminate the parameter t to write a Cartesian equation for the curve.

8.2. Write and evaluate an integral for the length of the curve over the interval t∈[0,1].

8.3. Suppose the equations had been $x(t) = 3e^t$ and $y(t) = 6e^t$, instead. The integral becomes easy. Why? Describe the parametric motion on the interval t∈[0,1].

9. (10 pts) Consider the vector $\vec{v} = \langle 4, -2, 0 \rangle$

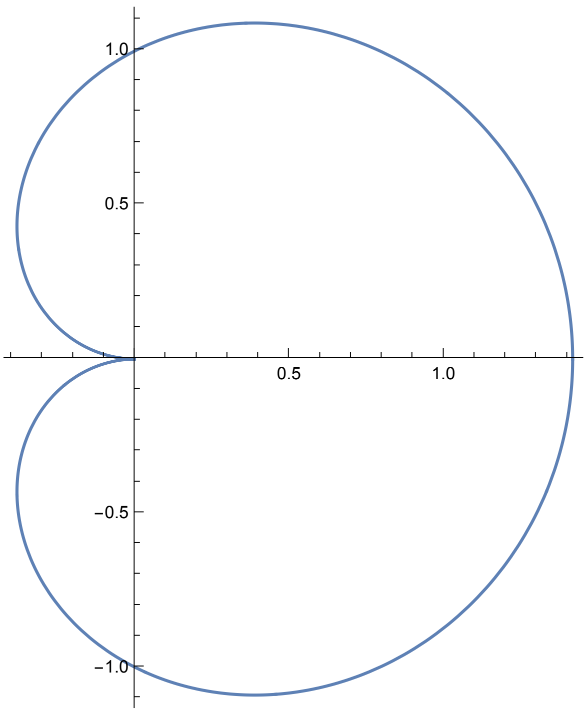
9.1. Compute the exact length of $\vec{v}$ (that is, the norm of $\vec{v}$, or $|\vec{v}|$).

9.2. Create a unit vector pointing in the direction of $\vec{v}$.

9.3. Find a vector $\vec{u}$ perpendicular to $\vec{v}$ (and **show** that it is perpendicular by computing the dot product).

9.4. Use the cross product to find a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

10. (10 pts) Consider the polar curve $r = \sqrt{\cos(\theta) + 1}$ (shown below)



10.1. What is the area enclosed by the curve on the interval $-\frac{\pi}{2} \leq \theta \leq \pi$?

10.2. Sketch the area we computed above, and estimate it (roughly). Does your estimate agree with your calculation?

11. **Extra Credit** (5 pts): Use the limit definition of the derivative ("the most beautiful idea in calculus") to compute the derivative of $f(x) = e^x$. You may use the fact that the base e was chosen because its graph has a slope of 1 at x=0.