Directions: All problems are equally weighted. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Note**: you **must** skip **one** of the eight problems (except for Problem 2). Write "skip" across it, so I'll easily know which one you are skipping. **Good luck!**

Problem 1 (10 pts) Let $f(x) = \dfrac{x^3}{x^2-1}$. **Justify your answers to the following**:

a. How would you describe the function f: is it in a particular class that you recognize? What classes of functions make it up?

f(x) is a rational function made up of a ratio of polynomial functions

b. What is its domain of definition? Where is it defined?

$x^2-1=0$

$(x+1)(x-1)=0$

$x=-1 \quad x=1$

(domain)
D = all numbers R except ± 1

It is defined on all numbers but -1 and 1, where the denominator will be equal to 0

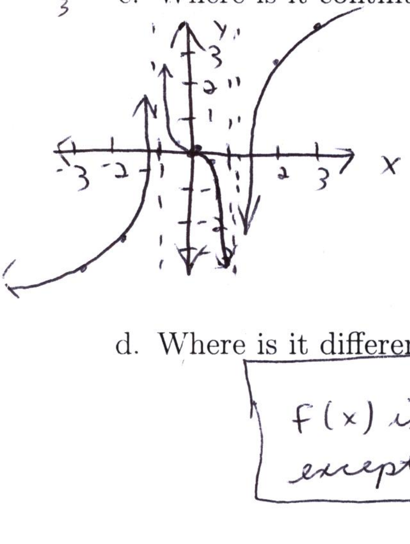
$\frac{27}{8}$

$\frac{-27}{8}$

.125

$\frac{8}{3}$

c. Where is it continuous? If it is discontinuous at a point, describe the type of discontinuity.



It is continuous on the intervals $(-\infty,-1)$ $(-1,1)$, and $(1,\infty)$

There is a jump at the points $x=1, x=-1$, where the one-sided limits are not equal.

X	Y
0	$-\frac{1}{8}$
2	$2\frac{2}{3}$
3	3.375
-2	

d. Where is it differentiable?

f(x) is differentiable on all its domain except ~~x = 1, -1~~ $x=1$ and $x=-1$

Problem 2 (10 points – you may not skip this one!) Consider the function $f(x) = \sqrt{x} - 2$.

a. (4 pts) Use the limit definition of the derivative to compute $f'(x)$.

$f(x) = \sqrt{x} - 2$

Limit def $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Substitute $\quad = \lim_{h \to 0} \frac{\sqrt{x+h} - 2 - (\sqrt{x} - 2)}{h}$

multiply by conjugate $\quad = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

x cancels out, and h cancels out ~~to~~ remove indeterminacy $\quad = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$

this is the quotient law

denominator is now composed of a sum of continuous functions (which are compositions of continuous functions) which means it is continuous, and 0 can be substituted for h

good

$= \frac{\lim_{h \to 0} 1}{\lim_{h \to 0} \sqrt{x+0} + \sqrt{x}}$

$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$ ✓

b. (3 pts) Use this derivative to write the equation of the tangent line to the function at $(1, -1)$.

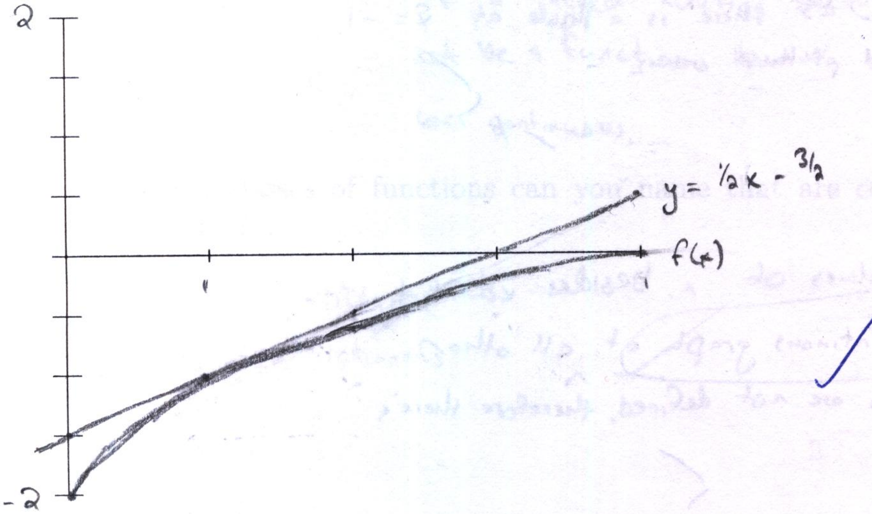
$f(1) = \sqrt{1} - 2 = 1 - 2 = -1$ ✓

$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2(1)} = \frac{1}{2}$

$y - (-1) = \frac{1}{2}(x - 1)$

$y = \frac{1}{2}(x-1) - 1 \rightarrow y = \frac{1}{2}x - \frac{3}{2}$ ✓

c. (3 pts) Graph both f and its tangent line below (label the y-axis from -2 to 2):



x	f(x)
0	-2
1	-1
2	-0.59
3	-0.27
4	0

✓

Nice work.

Problem 2 (10 points – you may not skip this one!) Consider the function $f(x) = \sqrt{x} - 2$.

a. (4 pts) Use the limit definition of the derivative to compute $f'(x)$.

$f(x) = \sqrt{x} - 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - 2 - (\sqrt{x} - 2)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} \cancel{-2} - (\sqrt{x} \cancel{+2})}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x + h \cancel{-x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$ ← substitute 0 in for h

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

b. (3 pts) Use this derivative to write the equation of the tangent line to the function at $(1, -1)$.

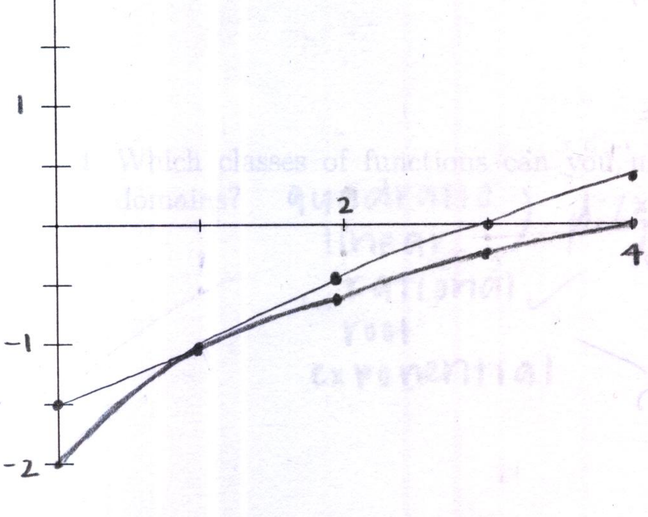
$m = f'(x) = \frac{1}{2\sqrt{x}}$

$= f'(1) = \frac{1}{2\sqrt{1}}$

$= \frac{1}{2}$

$y = -1 + \frac{1}{2}(x - 1)$

c. (3 pts) Graph both f and its tangent line below (label the y-axis from -2 to 2):



$f(x) = \sqrt{x} - 2$

tangent : $y = \frac{1}{2}x - \frac{3}{2}$

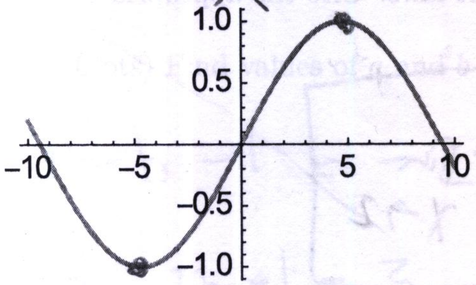
x	f(x)
0	-2
1	-1
2	-0.58
3	-0.26
4	0

x	y
0	-1.5
1	-1
2	-.5
3	0
4	.5

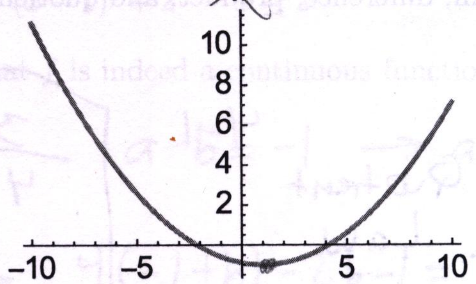
Nice work!

Problem 3 (10 points) Match the derivatives to the functions: the functions are on top (labelled I-IV) and their derivatives are below (labelled A-D). Give as many reasons as you can. Feel free to "decorate" the graphs as needed to explain.

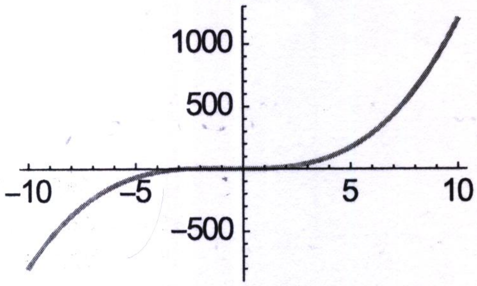
I



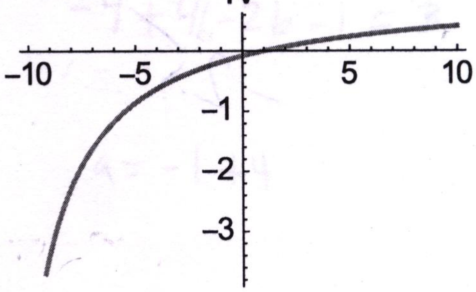
II



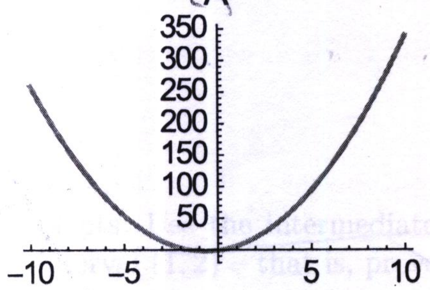
III



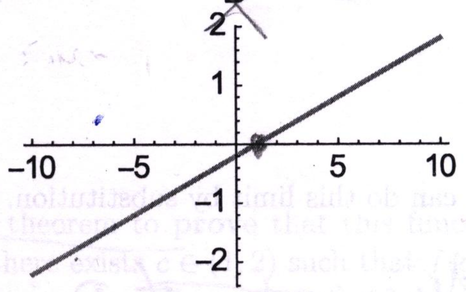
IV



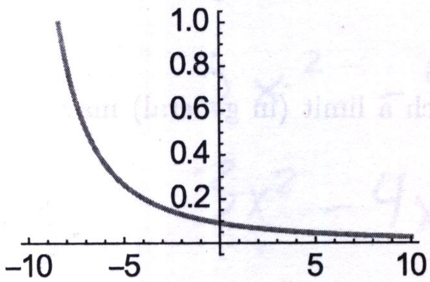
A



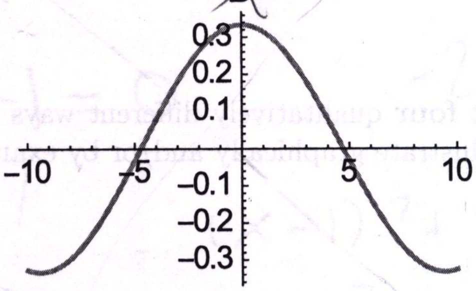
B



C



D



III + A: III seems cubic + odd, while A is a degree lower as a parabola + is even. III's slope appears to be zero around the origin which is where A is has y values around zero.

IV + C: IV start off with almost an infinitely positive slope. C reflects that, but IV also starts to get closer to zero + C gets closer to the x axis.

I + D: I is odd + D is even. The slope at x = 5 + -5 is zero, which is where D intercepts the x axis.

II + B: II is a parabola, + B is linear, so it's one degree down. At II's vertex is where the slope is zero + at that x coordinate is where B has an x intercept.

Problem 4 (10 points) Let $f(x) = \dfrac{x^2-1}{x+2}$.

a. (4 pts) Find $\lim_{x\to 2} f(x)$, **given only** the constant limit law $\left(\lim_{x\to c} a = a\right)$, the identity law $\left(\lim_{x\to c} x = c\right)$, and the sum, difference, product, and quotient limit laws. Cite the appropriate limit laws as you go.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-1}{x+2} \quad \text{quotient law} \quad = \frac{\lim_{x\to 2} x^2-1}{\lim_{x\to 2} x+2} \quad \text{sum law}$$

$$= \frac{\lim_{x\to 2} x^2-1}{\lim_{x\to 2} x + \lim_{x\to 2} 2} \quad \text{constant law} \quad = \frac{\lim_{x\to 2}(x^2-1)}{\lim_{x\to 2} x + 2} \quad \text{identity law} \quad = \frac{\lim_{x\to 2} x^2-1}{2+2} \quad \text{sum law}$$

$$= \frac{\lim_{x\to 2} x^2 - \lim_{x\to 2} 1}{4} \quad \text{constant law} \quad = \frac{\lim_{x\to 2} x^2 - 1}{4} \quad \text{product law} \quad = \frac{\lim_{x\to 2} x \cdot \lim_{x\to 2} x - 1}{4} \quad \text{identity law}$$

$$= \frac{2\cdot 2-1}{4} = \frac{4-1}{4} = \frac{3}{4}$$

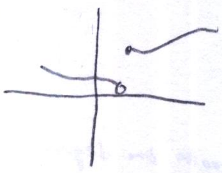
good

b. (2 pts) Explain why you can do this limit by substitution.

This is a rational function composed of two polynomials, polynomials are continuous and the division of two continuous functions is continuous on its domain. The domain is all real except $x=-2$. Since $x=2$ is in the domain of this continuous function, substitution for the limit $x\to 2$ is allowed.

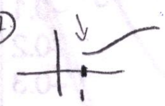
c. (4 pts) Describe at least **four** qualitatively different ways in which a limit (in general) may fail to exist. You may illustrate graphically and/or by examples.

(1)



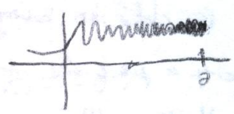
jump discontinuity
- left and right limits dont agree

(2)



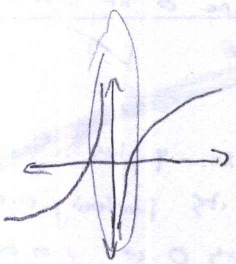
fails to exist at one because there is no limit $x\to 1^-$ - only one side

(3)



limit fails to exist because it never settles and approaches a single number

(4)



at $x=0$

The left and right limits dont agree

Well

Problem 5 (10 points) The function f is defined piecewise by three different quadratic functions stitched together – at least they're supposed to be stitched together – into a continuous function.

$$f(x) = \begin{cases} x^2 - 3 & x < 1 \\ ax^2 - bx - 1 & 1 \le x < 2 \\ (x-1)^2 + 2 & x \ge 2 \end{cases}$$

a. (6 pts) Find values of a and b so that f is indeed a continuous function.

$f(1) = (1)^2 - 3$ $\quad (x^2 - 3)$

$\boxed{f(1) = -2}$

$f(2) = (2-1)^2 + 2$ $\quad (x-1)^2 + 2$

$f(2) = 3$

$a(1)^2 - b(1) - 1 = -2$

$a - b = -1$

$a(2)^2 - b(2) - 1 = 3$

$4a - 2b = 4$

$a - b = -1$

$4a - 2b = 4$

$a = -1 + b$

$4(-1 + b) - 2b = 4$

$4b - 4 - 2b = 4$

$\frac{2b}{2} = \frac{8}{2}$ 4

$a = -1 + b$

$a = -1 + 4$

$a = 3$

$\boxed{f(2) = 3}$

$f(1) = -2$

$f(2) = 3$

$a = 3$

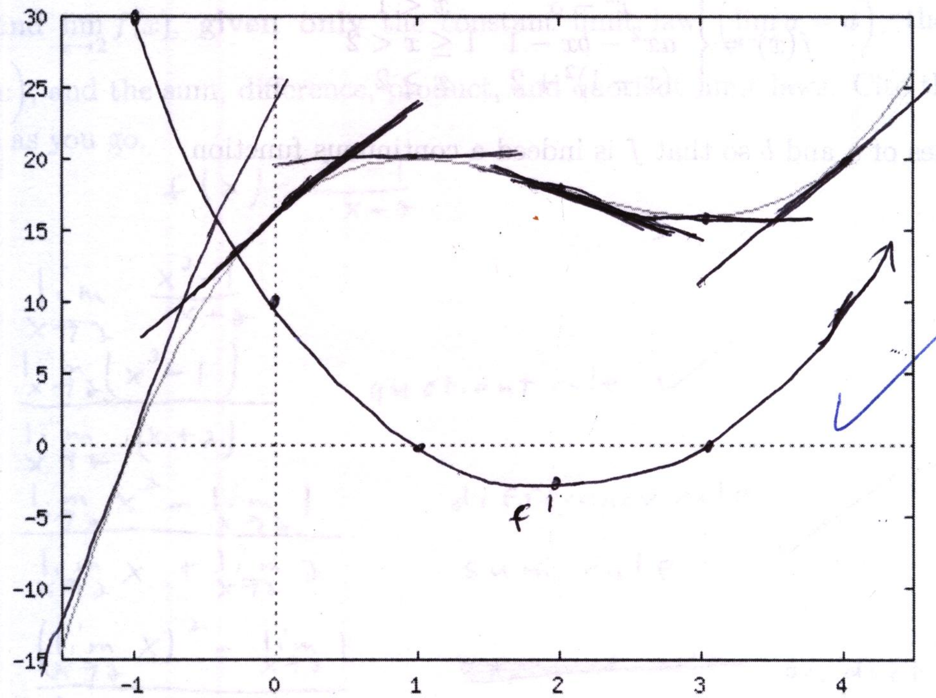
$b = 4$

b. (4 pts) Use the intermediate value theorem to **prove** that this function has a root on the interval $(1, 2)$ – that is, prove that there exists $c \in (1, 2)$ such that $f(c) = 0$.

The intermediate value theorem states that suppose f is continuous on an interval [a, b] and that a number N exists between f(a) and f(b), then there exists a number f(c) = N. So from this equation, the interval is (1, 2) and the number between the two is zero.

-1

Problem 6 (10 points) Consider the following graph of a function f:



$\frac{10}{1}$

$\frac{15}{.5}$

a. (6 pts) Draw in tangent lines to the curve at the points $x = -1, 0, 1, 2, 3, 4$, and give your estimates for their slopes in the table below:

Table 1: Fill in your slopes here (do your calculations for the slopes to the right of the table):

x	m
-1	30
0	10
1	0
2	-3
3	0
4	10

$\frac{15}{.5} = 30$ $\quad \frac{0}{1} = 0$ $\quad m = \frac{\text{rise}}{\text{run}}$

$\frac{10}{1} = 10$ $\quad \frac{10}{1} = 0$

$\frac{0}{2} = 0$

$\frac{-3}{1} = -3$

Nice work

b. (4 pts) Use estimates of the slopes of the tangent lines obtained above to construct a graph of the derivative function $f'(x)$. Draw it on the graph of f, using the same scale.

When you're done, explain why your graph makes sense.

It makes sense because it is parabolic, (degree 2) and the original was of degree 3.
Also, the derivative graph has zeros at the ~~pts~~ points that were minimums and maximums of the ~~gp~~ original.

Problem 7 (10 points) If a ball is thrown into the air from a height of 2 meters with a velocity of 30 m/s, its height (in meters) after t seconds is given by

$$s(t) = 2 + 30t - \frac{9.81}{2}t^2$$

$(2+h)(2+h)$

$4+4h+h^2$

a. (6 pts) Find its velocity when $t = 2$ seconds, using the limit definition of the derivative.

$$s'(2) = \lim_{h\to 0} \frac{2 + 30(2+h) - \frac{9.81}{2}(2+h)^2 - \left(2 + 30(2) - \frac{9.81}{2}(2)^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2+60} + 30h - 4.905(4+4h+h^2) - (\cancel{62} - 19.62)}{h}$$

$$= \lim_{h\to 0} \frac{30h - 4.905(4+4h+h^2) + 19.62}{h}$$

$$= \lim_{h\to 0} \frac{30h \cancel{-19.62} - 19.62h - 19.62h^2 \cancel{+19.62}}{h}$$

$$= \lim_{h\to 0} \frac{10.38h - 19.62h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(10.38 - 19.62h)}{\cancel{h}}$$

$$= 10.38 - 19.62(0)$$

$$= \boxed{10.38\ m/s}$$

b. (4 pts) Write the equation of the tangent line to the graph of s at $t = 2$ seconds.

$t = 2$

$2 + 30(2) - \frac{9.81}{2}(2)^2$

$62 - 19.62$

$= 42.38\ m$

$y - 42.38 = 10.38(x-2)$

$\boxed{y = 10.38(x-2) + 42.38}$

$x = 2$ seconds

$y = 42.38\ m$

Nice

Problem 7 (10 points) If a ball is thrown into the air from a height of 2 meters with a velocity of 30 m/s, its height (in meters) after t seconds is given by

$$s(t) = 2 + 30t - \frac{9.81}{2}t^2$$

$-19.62 + 60 + 2 = 42.38$

a. (6 pts) Find its velocity when $t = 2$ seconds, using the limit definition of the derivative.

$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ $(x+h)(x+h)$ $(x^2 + xh + xh + h^2)$

$\frac{-9.81}{2}t^2 + 30t + 2$

$-9.81t + 30 = f'(x)$

$= \frac{\left(\frac{-9.81}{2}(x+h)^2 + 30(x+h) + 2\right) - \left(\frac{-9.81}{2}x^2 + 30x + 2\right)}{h}$

$\frac{\left(\frac{-9.81}{2}x^2 - 9.81xh - \frac{9.81}{2}h^2\right) + 30x + 30h + 2 + \frac{9.81}{2}x^2 - 30x - 2}{h}$

$= \frac{-9.81xh - \frac{9.81}{2}h^2 + 30h}{h}$

$= \lim_{h \to 0} -9.81x - \frac{9.81}{2}h + 30 = -9.81x + 30 = -9.81t + 30$

b. (4 pts) Write the equation of the tangent line to the graph of s at $t = 2$ seconds.

$(2, 42.38)$

$-9.81(2) + 30 = 10.38$

$Y - Y_0 = m(x - x_0)$ $Y = 10.38(x-2) + 42.38$

good

— —

Problem 8 (10 points) Short answer:

a. Write a technically correct definition of a function.

A function is a rule that assigns each element x in a set D exactly one element (usually $f(x)$) in a set E.

good ✓

b. What is the most important definition in calculus, according to Prof. Long?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

✓

c. True or False (and explain!): Differentiability implies continuity, and continuity implies differentiability.

False! Differentiability does in fact imply continuity and smoothness, continuity does NOT imply differentiability. Continuity means $\lim_{x\to a} f(x) = f(a)$ for all x values. So the limit must exist and the value must exist and they must be equal. That means cusps are still continuous. Cusps are abrupt changes in rates and because of that not differentiable. Differentiability means a derivative can be taken there, which is the slope of the line tangent. There is no tangent line for a cusp. So a function can be continuous and not differentiable

no tangent

d. Which classes of functions can you name that are continuous and differentiable on their domains?

Polynomials → power

Rational ← not quite – no implication. ✓

exponentials

root

compositions (if $f(g(x))$ if f + $g(x)$ is continuous and differentiable)

reciprocal

10

Problem 8 (10 points) Short answer:

a. Write a technically correct definition of a function.

The function ~~or~~ a function is a rule that assigns to each element x in a set D, one element called ($f(x)$) in a set E.

unique

b. What is the most important definition in calculus, according to Prof. Long?

The most important definition in calculus is the derivative function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

c. True or False (and explain!): Differentiability implies continuity, and continuity implies differentiability.

This is false. However, it is true that differentiability implies continuity, but continuity doesn't imply differentiability. If f is differentiable at a, then f is continuous at a. However if f is continuous at a, this doesn't mean that f is differentiable at a.

d. Which classes of functions can you name that are continuous and differentiable on their domains?

- Polynomials.
- Rational Functions.
- Trigonometric functions.
- Exponential functions.

Nice work