Wallpaper Groups: Alhambra, Escher and Symmetries

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October 24, 2021

1 Introduction

The Alhambra is a palace in Granada, Spain renowned for its decorative geometric patterns and tilings [6]. The palace itself is conjectured to contain examples of all 17 "wallpaper patterns," where each is a unique way to tile a plane. In other words, a wallpaper pattern covers a two-dimensional region with no gaps or overlaps using copies of the same shapes. The graphic artist M.C. Escher's visits to the temple inspired a collection of his art pieces known as Escher's tilings (Fig. 1).

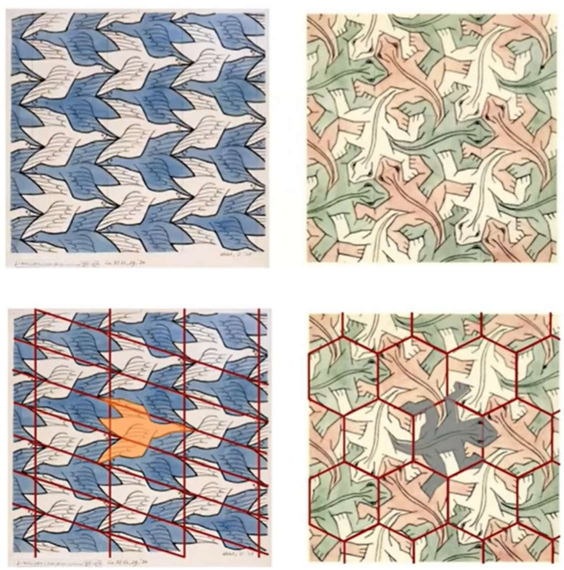


Figure 1: Two tilings by M.C. Escher with underlying grid [11].

Notably, if we look closer at the tilings in Figure 1, we notice an underlying grid of polygons. Based on mathematical research at the time—primarily Polya's work [8]—Escher's tilings were created from wallpaper patterns, where each pattern is composed of symmetries of a single base shape arranged to fill the plane.

While it may seem like there are infinitely many ways to tile the plane, there are in fact, only 17 distinct wallpaper patterns in two dimensions, each defined by a unique group of symmetries. For example, while the two tilings shown below may appear to be completely different, they are classified within the same wallpaper group.

Figure 2: Two patterns generated by the same wallpaper group [3].

In this paper, we will explore the mathematical classification of wallpaper groups. To do this, we first define the notion of symmetry groups in the context of group theory. Using this, we introduce several types of symmetry groups, namely cyclic groups, dihedral groups, and frieze groups, to motivate a definition for wallpaper groups.

2 Symmetry Groups

Before introducing types of symmetry groups, we will first define the notion of a symmetry and how it relates to the abstract concept of groups.

2.1 Symmetries

We begin by discussing four types of transformations (Fig. 3).

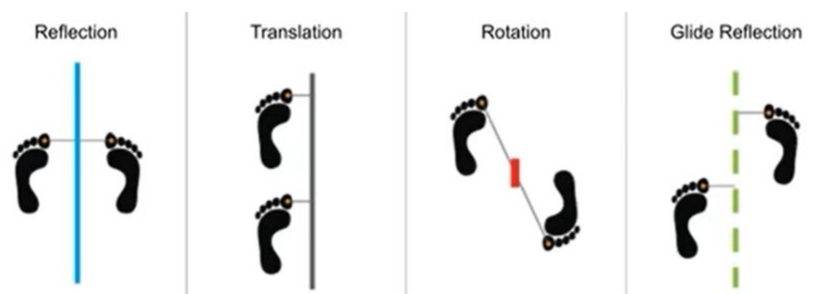


Figure 3: Illustrating a reflection, translation, rotation, and glide reflection [11].

Definition 2.1.1. The movement of a figure around a given fixed point is a *rotation*.
Definition 2.1.2. A *reflection*, also called a"*flip*" is a mirror of the figure over a given line.
Definition 2.1.3. A *translation* moves every point of the figure by the same distance in a given direction.
Definition 2.1.4. A *glide reflection* is a translation combined with a reflection across an axis parallel to the direction of translation.

These definitions permit us to formally characterize the symmetries of any figure.

Definition 2.1.5. A *symmetry* is a transformation of the figure under which the figure is invariant.

In other words, a symmetry is a rigid motion of the figure where taking a copy of the figure, moving it in space, and placing the copy back on the original figure would exactly cover it.

2.2 Groups of Symmetries

We next introduce fundamental notions of group theory.

Definition 2.2.1. A *group* is an ordered pair $(G, \star)$, where G is a set and $\star$ is a binary operation on G that follows the following axioms:
i) The binary operation $\star$ is associative such that for all $a, b, c \in G$, we have $(a \star b) \star c = a \star (b \star c)$.
ii) There exists an element $e \in G$ such that for all a $\in G$, we have $a \star e = e \star a = a$. We call this the *identity* of G.
iii) For all $a \in G$, there is an element a^{-1}, called an *inverse*, such that $a \star a^{-1} = a^{-1} \star a = e$.

We can now define a group in terms of a figure's symmetries.

Definition 2.2.2. A *symmetry group* G is a set of all symmetries of a shape under the binary operation of composition of transformations.

2.3 Cyclic Groups

To illustrate the concept of symmetry groups, we introduce cyclic groups.

Definition 2.3.1. A group H is *cyclic* if H can be generated by a single element. In other words, H is cyclic if there is an element $x \in H$ such that $H = \{x^n \mid n \in \mathbf{Z}\}$. The cyclic group H generated by x is written as $H = \langle x \rangle$.

The group of rotations of an n-gon forms a cyclic group of order n, denoted C_n. $C_n = \{1, R^1, R^2, ..., R^n - 1\}$ where R^i is a rotation of $\frac{2\pi i}{n}$ radians (Table 1).

2.4 Dihedral Groups

Definition 2.4.1. For each $n \in \mathbf{Z}^+$, $n \geq 3$, the *dihedral group* D_n is the group of symmetries of a regular n-gon under composition of transformations.

Lemma 2.4.2. *For each $n \in \mathbf{Z}^+$, we have $|D_n| = 2n$.*

Proof. Define an n-gon Q with vertices numbered $\{1, 2, 3, ..., n\}$ in mod n. Because symmetries are rigid motions, the position of any two vertices will determine the position all other vertices in the n-gon; thus, we will only consider how the symmetry affects vertices 1 and 2. Given any vertex i, there is a rotation p that sends vertex 1 to vertex i. Vertex 2 is adjacent to vertex 1, so under rotation p, vertex 2 is sent to either vertex $i+1$ or $i-1$. By composing p with a reflection across the line through vertex i and the center of the n-gon, vertex 2 can be sent to vertex $i+1$ or $i-1$. Thus, by applying symmetries, there are $2n$

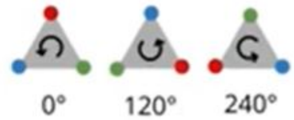
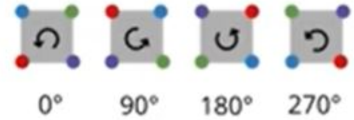
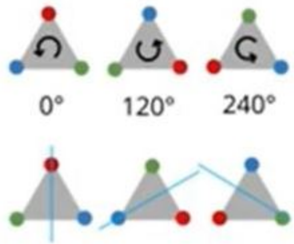
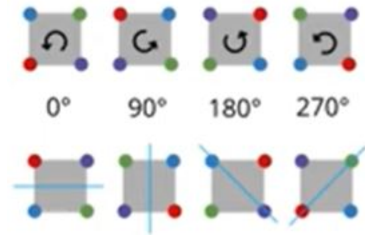
Symmetries	$n = 3$	$n = 4$
C_n Rotations		
D_n Rotations and Reflections		

Table 1: The elements of C_n is a subset of the elements of D_n [11].

positions where vertex 1 and 2 can end up, so $|D_n| = 2n$. Thus, any n-gon has exactly $2n$ symmetries [2]. □

In particular, these symmetries are the n rotations about the center of $\left(\frac{2\pi}{n}\right) i$, for $1 \leq i \leq n-1$, which can be characterized by the group C_n, as well as the n reflections through the n lines of symmetry. If n is odd, each line of symmetry passes through a vertex and the midpoint of the opposite side, and if n is even, there are $\frac{n}{2}$ lines of symmetry through two opposite vertices and $\frac{n}{2}$ lines of symmetry that perpendicularly bisect two opposite sides (Table 1).

Remark. The dihedral group D_n is often written as D_{2n} where the subscript gives the order of the group rather than the number of vertices of the n-gon.

2.5 Frieze Groups

The symmetry groups discussed so far are composed of rotations and reflections and have the property that the symmetries fix the center of the figure. Removing this constraint, we can add translation and glide reflections. Before generating wallpaper groups in two dimensions, we will first consider these transformations in one dimension which yields a frieze pattern (Fig. 4).

Definition 2.5.1. A *frieze pattern* is a two-dimensional design that repeats in only one direction.
Definition 2.5.2. A *frieze group* is the set of symmetries of a *frieze pattern*.

Theorem 2.5.3. There are exactly 7 frieze groups [12].

Each of the frieze groups is characterized by a unique set of symmetries (Fig. 5).

Figure 4: Examples of frieze patterns [5].

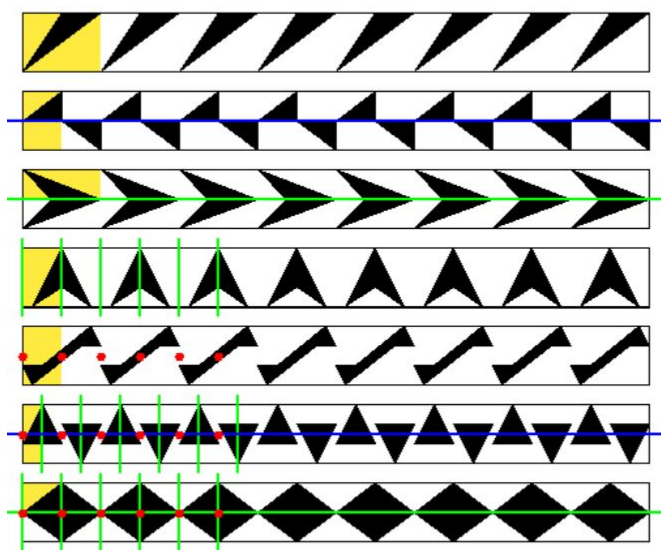


Figure 5: The 7 Frieze Groups. Reflections are denoted by green lines, blue lines indicate glide reflections, red dots indicate order 2 rotation centers [10].

2.6 Wallpaper Groups

Frieze patterns are constrained to repeating in one direction. With the notion of a pattern repeating in two directions, we can characterize a wallpaper pattern. Examples of each of the 17 wallpaper patterns are shown below (Fig. 6).

Definition 2.6.1. A *wallpaper pattern* is a two-dimensional design that repeats in two directions.

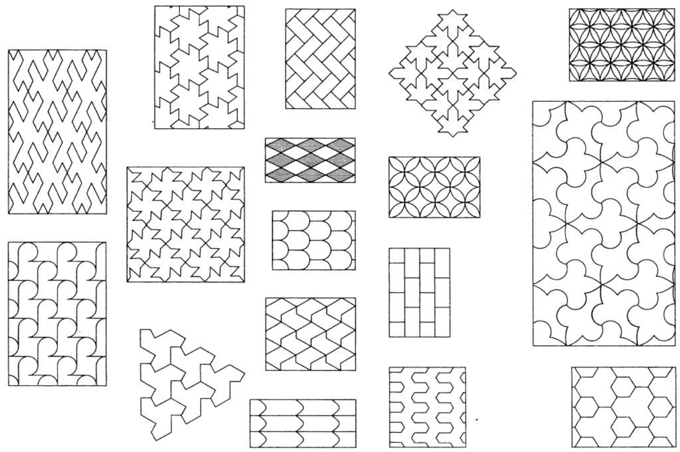


Figure 6: Examples of each of the 17 wallpaper groups [1].

Definition 2.6.2. A *wallpaper group* is the set of symmetries of a *wallpaper pattern*.

Theorem 2.6.3. There are exactly 17 wallpaper groups [9].

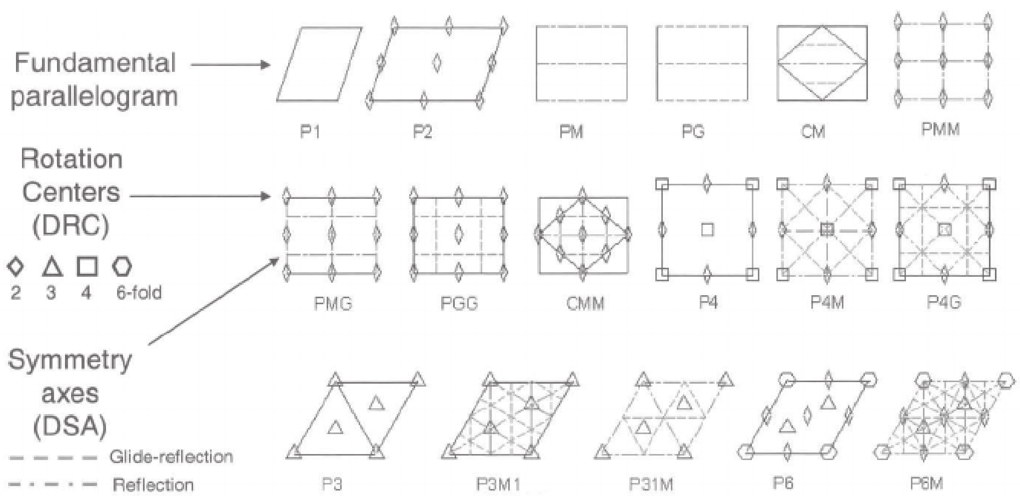


Figure 7: 17 wallpaper groups with symmetries shown [4].

Observe that each wallpaper group is identified by a string of two to four letters and numbers that describe its symmetries (Fig. 7).

Notation. A primitive cell, denoted "p," is the fundamental region of the symmetry group, in other words, the minimal region repeated in the pattern. This is followed by a number in the name indicates the highest order of rotational symmetry (none, 2-fold, 3-fold, 4-fold, 6-fold). All but two wallpaper groups are described in terms of primitive cells, and the remaining two are described in terms of centered cells, denoted by "c," that are larger than the primitive cell. Thus, there is internal repetition in centered cells, and the sides of a centered cell are tilted in relation to a primitive cell. If present, the next symbols indicate symmetries relative to one translation axis of the pattern, called the "main" axis. An "m" denotes a "mirror" or reflection, and a "g" denotes a glide reflection. The first of these letters indicates a reflection or glide reflection perpendicular to the "main" axis. However, the second "m" or "g" indicates a transformation parallel or tilted at an angle of $\frac{\pi}{n}$ relative to the "main" axis for some $n > 2$.

To identify the wallpaper group of a given design, we can use the following flowchart (Fig. 8).

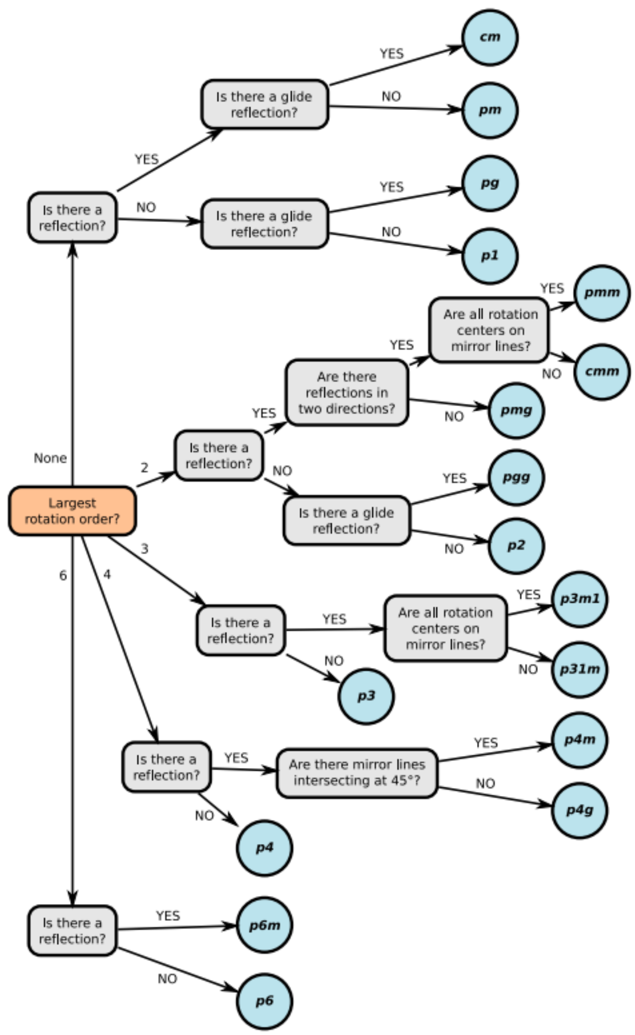


Figure 8: Flowchart to identify an image's wallpaper group classification [7].

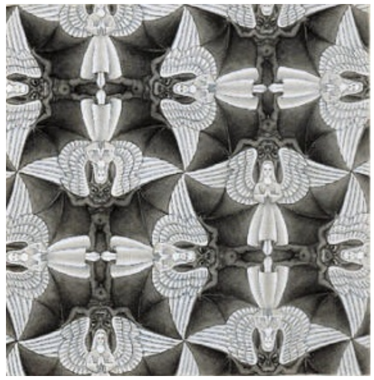


Figure 9: Escher Sketch 45 “Angels and Devils” [7].

Example 2.6.4. *What is the wallpaper group for Escher's Sketch 45 “Angels and Devils”* (Fig. 9)*?*

The largest rotation order in this sketch is 4 with centers at the points where the wings of four angels and four devils intersect. There are both horizontal and vertical reflections axes bisecting the angels and devils; however there are no mirror lines intersecting at 45° because the only reflection axes are horizontal and vertical, the wallpaper group of this pattern is p4g [7].

Acknowledgements

I would like to thank the MIT PRIMES Circle program for providing me with the resources and guidance to dive into complex math topics. In particular, I would like to thank Peter Haine for editing my work. Most of all, I would like to thank my mentor Kaili Liu for supporting my learning and guiding me through each step of the process this semester.

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