Section 1.2: Propositional Logic

January 22, 2025

Abstract

Now we're going to use the tools of formal logic to reach logical conclusions ("prove theorems") based on wffs formed by some given statements. This is the domain of **propositional logic**.

- **Propositional wff**: represent some sort of argument, to be tested, or proven, by **propositional logic**.

- **valid arguments**, e.g.

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$$

have **hypotheses** (we suppose – assert – that the P_i are true), and a **conclusion** (Q). To be *valid*, this argument must be a tautology (always true). To be an *argument*, Q must not be identically true (i.e. a fact, in which case the hypotheses are **irrelevant**, by the truth table of implication).

Examples

a. Valid argument: If logic is hard, then I am a monkey's uncle. I am not a monkey's uncle. Therefore, logic is not hard.

b. Invalid arguments: If people are crazy, then they should be in an asylum. I am in an asylum.

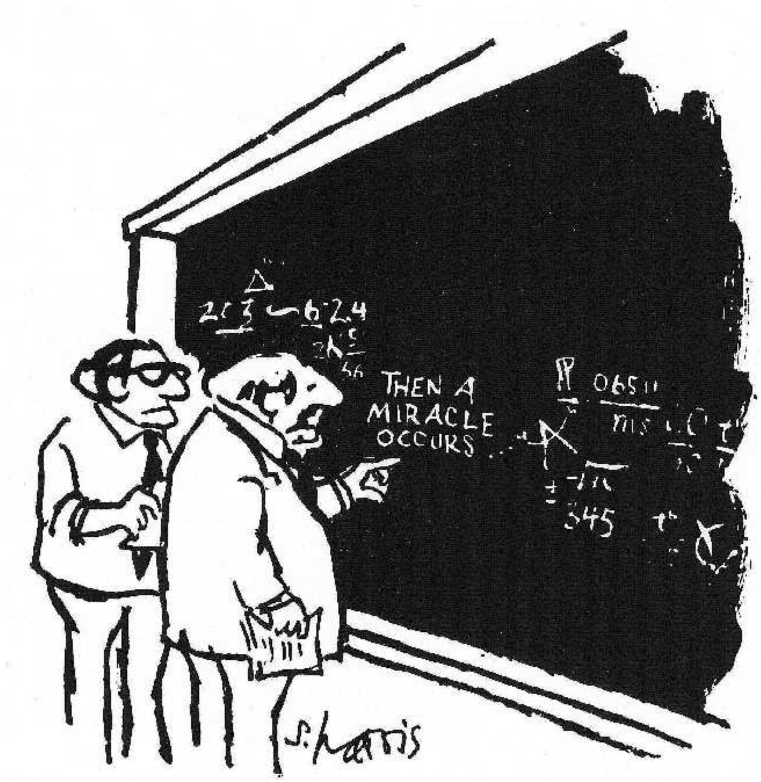
   i. Therefore $1 + 1 = 2$.

   ii. Therefore I am crazy.

- **Proof Sequence**: a sequence of wffs in which every wff is an hypothesis or the result of applying the formal system's derivation rules (truth-preserving rules) in sequence.

Our Objective: to reach the conclusion Q from the hypotheses P_1, P_2, …, P_n (with reasons!).

The famous cartoonist Sidney Harris knows that you need to follow the rules (equivalence and inference rules) in a proof sequence:



"I think you should be more explicit here in step two."

- Types of derivation rules:
  - **Equivalence rules**: We can **substitute** equivalent wffs in

TABLE 1.11

Equivalence Rules		
Expression	**Equivalent to**	**Name/Abbreviation for Rule**
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative—comm
$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	Associative—ass
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	De Morgan's Laws—De Morgan
$P \rightarrow Q$	$P' \vee Q$	Implication—imp
P	$(P')'$	Double negation—dn
$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$	Definition of equivalence—equ

a proof sequence by invoking equivalence rules.

You're asked to prove the implication equivalence rule in Practice 9: that is, prove that

$$P \to Q \longleftrightarrow P' \vee Q$$

is a tautology (notice how we use order of precedence of operations to drop parentheses). How would you prove it? And can you see another theorem, suggested by the commutativity of disjunction, as well as double negation (i.e. $Q \longleftrightarrow (Q')')$?

Implication seems somewhat unusual, but it is suggested by Exercise 11a, section 1.1:

"If the food is good, then the service is excellent."

So when we negate it, it leads to the rule

$(P \to Q)' \iff P \wedge Q'$.

– **Inference rules**: from given hypotheses, we can deduce certain conclusions.

TABLE 1.12

Inference Rules		
From	**Can Derive**	**Name/Abbreviation for Rule**
$P, P \to Q$	Q	Modus ponens—mp
$P \to Q, Q'$	P'	Modus tollens—mt
P, Q	$P \wedge Q$	Conjunction—con
$P \wedge Q$	P, Q	Simplification—sim
P	$P \vee Q$	Addition—add

* **modus ponens:** If P is true, and Q follows from P, then Q is true: $P \wedge (P \to Q) \to Q$

* **modus tollens:** If Q follows from P, and Q is false, then P is also false: $(P \to Q) \wedge Q' \to P'$

* **conjunction:** If P is true, and Q is true, then they're both true together: $P \wedge Q \to (P \wedge Q)$

* **simplification:** If both P and Q are true, then they're each true separately: $P \wedge Q \to P$, and $P \wedge Q \to Q$

* **addition:** If P is true, then either P or Q is true (or both): $P \to P \vee Q$

Of these, addition may seem a little odd: what do you gain by adding an arbitrary argument Q to an already true wff P into a logical or?

The difference between equivalence rules and inference rules is that equivalence rules are bi-directional (work both ways), whereas some inference rules are uni-directional (work in only one direction - e.g., simplification; this is what **inference** is all about:
"from this we can infer that" does not mean that "from that we can infer this".

Practice 10, p. 30 (and also suggests a reasonable step 4.)

1.	$(A \wedge B') \to C$	*hyp*
2.	C'	*hyp*

Let's start with #19, p. 37: Prove that the argument

$$(A' \to B') \wedge B \wedge (A \to C) \to C$$

is valid.
Then, for a more elaborate example, let's look at #31, p. 37, which shows that one can prove anything if one introduces a contradiction (e.g. this problem, on a mensa quiz I once found: "If 1/2 of 24 were 8, what would 1/3 of 18 be?"[1]). Also called an **inconsistency**, this is a beautiful and important example:

$$P \wedge P' \to Q$$

Notice that in the table 1.14 (More Inference Rules, p. 37) some rules appear twice (e.g. contraposition): two uni-directionals can make a bi-directional (which makes this effectively an **equivalence rule**).

Note for your homework: you are not allowed to invoke the rule that you are trying to prove! Notice that the entries in this table are followed by exercise numbers: it is in those exercises that the results are actually proved.

---

[1]Their solution (and I quote): "The best way to solve this is by setting up proportions: $\dfrac{1/2 \times 24}{8} = \dfrac{1/3 \times 18}{z}$. Then $\dfrac{12}{8} = \dfrac{6}{z}$, so $z = 4$." But wait – didn't they say that 1/2 of 24 is 8?!

TABLE 1.14		
More Inference Rules		
From	**Can Derive**	**Name/Abbreviation for Rule**
$P \to Q, Q \to R$	$P \to R$ [Example 16]	Hypothetical syllogism—hs
$P \vee Q, P'$	Q [Exercise 25]	Disjunctive syllogism—ds
$P \to Q$	$Q' \to P'$ [Exercise 26]	Contraposition—cont
$Q' \to P'$	$P \to Q$ [Exercise 27]	Contraposition—cont
P	$P \wedge P$ [Exercise 28]	Self-reference—self
$P \vee P$	P [Exercise 29]	Self-reference—self
$(P \wedge Q) \to R$	$P \to (Q \to R)$ [Exercise 30]	Exportation—exp
P, P'	Q [Exercise 31]	Inconsistency—inc
$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$ [Exercise 32]	Distributive—dist
$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$ [Exercise 33]	Distributive—dist

– **Deduction method**: if you seek to prove an implication (that is, if the consequent of the theorem is itself an implication), then you can simply add the antecedent of this consequent implication to the hypotheses making up the antecedent, and prove the consequent of the concluding implication:

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \to (R \to S)$$

can be replaced by

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R \to S$$

This is really "Exportation" (from Table 1.14) backwards, which means that Exportation is actually an equivalence rule (it is: check the truth tables, or construct a proof).

If you're interested in seeing why this rule works, you should try #55, p. 38, but think of it this way: we're interested in assuming that all the P_i are true, and see if we can deduce the implication $R \to S$. If R is false, then the implication is true. The only potentially problematic case is where R is true, and S is false. Then what we want to know is: given that

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R$$

are true, is S true?

Exercise #37, p. 38:

$$(A' \to B) \wedge (B \to C) \wedge (C \to D) \to (A' \to D)$$

- **Hypothetical syllogism**: The theorem above (#37) can be proven more easily using this method:

  $$(P \to Q) \wedge (Q \to R) \to (P \to R)$$

  (one of the rules in Table 1.14). This rule might well be referred to as **transitivity**.

A new rule is created each time we prove an argument; but we don't want to create so many rules that we keel over under their weight! Keep just a few rules in view, and learn how to use them well....

- Our goal may well be to turn a "real argument" into a symbolic one. This allows us to test whether the argument is sound (that is, that the conclusion follows from the hypotheses), without being distracted by the verbiage.

  Exercise #47, p. 38: If the ad is successful, then the sales volume will go up. Either the ad is successful or the store will close. The sales volume will not go up. Therefore the store will close. (A, S, C)

  The trouble comes because the language may obscure the structure of the syllogism; we seek to cut through the language to get at the logical skeleton and work with that.

- The propositional logic system is complete and correct:

  - **complete**: **every** valid argument is provable (we can show that it is a tautology).
  - **correct**: **only** a valid argument is provable (only tautologies are provable).

  The derivation rules are truth-preserving, so correctness is pretty clear; completeness is not! How can we tell if we can prove **every** valid argument?!