Generating (function) closed-form solutions for Fibonacci and Catalan sequences

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Abstract

In this little note I use generating functions to find closed-form solutions to two important recurrence relations (Fibonacci and Catalan numbers).

Rus May of Morehead State University loves generating functions more than anyone I know – thanks for the enthusiasm Rus – it's contagious!

1 Using Generating Functions

In this case we are using generating functions defined as series whose coefficients are terms in sequences that any self-respecting mathematician would want to know. We're looking for formulas for Fibonacci and Catalan numbers in this brief note, so we'll define

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

and

$$c(x) = \sum_{n=0}^{\infty} C_n x^n,$$

where f will lead us to the Fibonacci numbers ($\{F_n\}_{n\in\mathbf{N}^+}$), and c will lead to the Catalan numbers ($\{C_n\}_{n\in\mathbf{N}^+}$). Here N^+ denotes the natural numbers plus 0 – the "whole" numbers.

The strategy: cleverly concoct relationships these functions satisfy, choosing the relationships based on the recurrence relations which define each sequence. Once we have f and c as functions of x, we expand those functions in power series, and identify coefficients – and we'll have our closed-form solutions.

2 Fibonacci Numbers

The Fibonacci numbers have a "celebrated" reputation in Mathematics: "2,3,5,8, who do we appreciate? Fibonacci!" There's even a song by the Jackson Five that celebrates them: "Fibonacci – Easy as 1,1,2,3". I'm sure I've heard it somewhere....

Fibonacci numbers are defined recursively, as

$$\begin{cases} F_0 = 1 \\ F_1 = 1 \\ F_{n+1} = F_n + F_{n-1} \end{cases}$$

With $f(x) = \sum_{n=0}^{\infty} F_n x^n$, we write

$$f(x) = F_0 + \sum_{n=1}^{\infty} F_n x^n$$

and

$$xf(x) = x\sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_n x^{n+1} = \sum_{n=1}^{\infty} F_{n-1} x^n$$

Therefore

$$f(x) + xf(x) = F_0 + \sum_{n=1}^{\infty}(F_n + F_{n-1})x^n = F_0 + \sum_{n=1}^{\infty} F_{n+1} x^n$$

Let's multiply through both sides by an x:

$$x(f(x) + xf(x)) = F_0 x + \sum_{n=1}^{\infty} F_{n+1} x^{n+1}$$

Recognizing that last term as most of $f(x)$ (just missing the first few terms), we write

$$x(f(x) + xf(x)) = F_0 x + (f(x) - (F_0 + F_1 x)) = f(x) - F_0 = f(x) - 1$$

Whew! That was a lot of work, but now we can solve for $f(x)$: throwing all the $f(x)$ stuff to one side, we obtain

$$(x^2 + x - 1)f(x) = -1$$

or

$$f(x) = \frac{-1}{x^2 + x - 1} \tag{1}$$

so long as $x^2 + x - 1 \neq 0$. The roots of this polynomial equation are $r_1 = \frac{-1-\sqrt{5}}{2}$ and $r_2 = \frac{-1+\sqrt{5}}{2}$. If you're familiar with the "golden mean", $\gamma = \frac{1+\sqrt{5}}{2}$, then you'll notice that $r_1 = -\gamma$, and $r_2 = \frac{1}{\gamma}$.

Now this is where the trick comes in: we expand the rational function in Eq. (1) as a power series, and pick off the coefficients as the Fibonacci numbers. What a cool trick! First up, though, is a partial fraction decomposition:

$$f(x) = \frac{-1}{(r_1 - x)(r_2 - x)} = \frac{1}{\sqrt{5}}\left(\frac{-1}{r_1 - x} + \frac{1}{r_2 - x}\right) \tag{2}$$

and then we use series: since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

we obtain

$$\frac{1}{r-x} = \frac{1}{r}\left(\frac{1}{1-\frac{x}{r}}\right) = \frac{1}{r} + \frac{x}{r^2} + \frac{x^2}{r^3} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{r^{n+1}}$$

we use this result twice in Eq. (2) to write

$$f(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{-1}{r_1^{n+1}} + \frac{1}{r_2^{n+1}} \right) x^n$$

We can do a little algebra (and use what we know about r_1 and r_2) to uglify the expression a little – combining denominators, etc.:

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{-r_2^{n+1} + r_1^{n+1}}{\sqrt{5}(-1)^{n+1}} \right) x^n = \sum_{n=0}^{\infty} \left(\frac{-\gamma^{-(n+1)} + (-\gamma)^{n+1}}{\sqrt{5}(-1)^{n+1}} \right) x^n$$

Finally, after all that, good lord, we can write

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{-(-\gamma)^{-(n+1)} + \gamma^{n+1}}{\sqrt{5}} \right) x^n$$

so that the n^{th} Fibonacci number F_n is given by

$$F_n = \frac{\gamma^{n+1} - (-\gamma)^{-(n+1)}}{\sqrt{5}}$$

It's hard to believe that those are integers, but it's true! Furthermore, since $\gamma \approx 1.618$, the second term becomes negligible as n gets big:

$$F_n \approx \frac{\gamma^{n+1}}{\sqrt{5}}$$

So, for example, $F_{20} = 10946$, and so is $\frac{\gamma^{21}}{\sqrt{5}}$ (at least according to Mathematica). It's not quite 10946, since there's a little error in there – but it's not off by much: $\frac{-(-\gamma)^{-21}}{\sqrt{5}} \approx 0.0000182715$.

In fact, if you'll simply round the expression $\frac{\gamma^{n+1}}{\sqrt{5}}$, it generates every Fibonacci number correctly! Amazing....

Now, to be honest, I usually index my Fibonacci numbers from 1, so that $F_1 = F_2 = 1$, and the rest is history. It's easy to adjust our formulas, if you do too (and, in fact, they look more elegant – which is actually important to mathematicians). We just shift n by 1:

$$F_n = \frac{\gamma^n - (-\gamma)^{-n}}{\sqrt{5}} \tag{3}$$

and

$$F_n \approx \frac{\gamma^n}{\sqrt{5}}$$

One advantage of doing so is that when you continue backwards to $-\infty$ with the Fibonacci formula (Eq. (3) – and I don't think Fibonacci ever did), you get a lovely symmetry:

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_n	-8	5	-3	2	-1	1	0	1	1	2	3	5	8

That is, odd-indexed Fibonacci numbers are equal, whereas even-indexed Fibonaccis are negatives of each other:

$$F_{-2n} = -F_{2n}$$
$$F_{-(2n+1)} = F_{2n+1}$$

3 Catalan Numbers

I was inspired to write this little note by the following problem, which I chose for my discrete math final exam, spring 2020 (and initially discovered at this website, from which the following figure was taken):

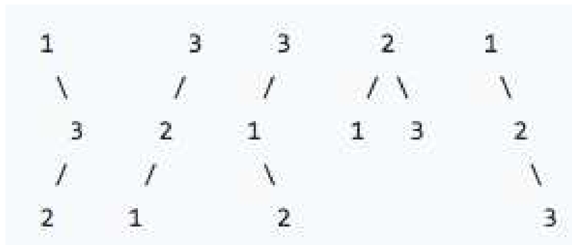

Figure 1: I don't know if the person who created this figure was being disingenuous, or just careless – but it might have helped to emphasize the symmetry this problem possesses.

Problem: Given $n \in \mathbb{N}$, find C_n – the number of structurally unique binary search trees that store values 1 through n. (For convenience I defined $C_0 = 1$.) Figure 1 shows that $C_3 = 5$.

Using the same quaint graphing technique, but emphasizing the symmetry, we see that there are five possible BSTs for $n = 3$:

```
1         1         2         3         3
 \         \       / \       /         /
  3         2     1   3     2         1
 /           \             /           \
2             3           1             2
```

Or maybe this representation is more suggestive (with the $n = 2$ and $n = 1$ cases thrown in for good measure):

```
1 1          2          3 3
 \ \        / \        / /
  3 2      1   3      2 1                    2   1
 /   \               /   \                  /     \
2     3             1     2                1       2             1
```

Maybe then it is becomes clearer that

$$C_3 = C_0 * C_2 + C_1 * C_1 + C_2 * C_0$$

Partition $\{1, 2, 3\}$, and created rooted sub-trees, based on the root's relative position:

1:	(nothin' to the left) * (two to the right)	$\{\}$ and $\{2,3\}$
2:	(one to the left)*(one to the right)	$\{1\}$ and $\{3\}$
3:	(two to the left)*(nothin' to the right)	$\{1,2\}$ and $\{\}$

So we multiply the number of ways of arranging $i-1$ things to the left of x_i times the number of ways of arranging $n-i$ things to the right; and so it goes....

After some considerations like these, of the recursive nature of the problem, I wrote down the recurrence relation. Only later did I discover that Catalan numbers are defined recursively in this same way:

$$\begin{cases} C_0 = 1 \\ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \end{cases}$$

With that recurrence relation in mind, we notice that if we square the generating function $c(x) = \sum_{n=0}^{\infty} C_n x^n$, we get

$$c(x)^2 = \sum_{i=0}^{\infty} C_i x^i \sum_{j=0}^{\infty} C_j x^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_i C_j x^{i+j};$$

then, for a given power n of x we should combine all the terms such that $n = i + j$, and then re-write the double sum as a single series (and use the recurrence relation to simplify):

$$c(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n = \sum_{n=0}^{\infty} \left(C_{n+1} \right) x^n.$$

Multiplying both sides by x, we get

$$xc(x)^2 = \sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=1}^{\infty} C_n x^n = c(x) - C_0 = c(x) - 1$$

Zut, alors! We've discovered that

$$xc(x)^2 - c(x) + 1 = 0.$$

We can solve this simple quadratic equation for $c(x)$, to obtain

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

and we seek to write $c(x)$ as a power series. However, it's simpler to find a power series for

$$2xc(x) = 1 \pm \sqrt{1-4x}$$

In the limit as $x \to 0$, we must have $c(x) = C_0 = 1$. Only the choice of the minus sign admits this limit, so we can pitch the plus, and need only consider the form

$$2xc(x) = 1 - \sqrt{1-4x}.$$

We use the generalized binomial theorem to write the power series on the right: starting from

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{1!}\frac{1}{2}x + \frac{1}{2!}\frac{1}{2}\left(\frac{1}{2}-1\right)x^2 + \frac{1}{3!}\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^3 + \dots,$$

or

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-(n-1)\right)x^n \equiv \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n,$$

we write our particular expression as

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} (-4)^n \binom{\frac{1}{2}}{n} x^n \equiv \sum_{n=0}^{\infty} s_n x^n$$

(and define the coefficients s_n at the same time). We note that

$$s_n = (-4)^n \binom{\frac{1}{2}}{n} = (-4)^n \frac{1}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-(n-1)\right) = \frac{(-2)^n}{n!}(1-0)\,(1-2)\ldots(1-2(n-1))$$

There are n terms in the product, but only $n-1$ of them are negative. Hence

$$s_n = \frac{(-2)^n}{n!}\,(1(-1)(-3)\ldots(3-2n)) = -\frac{2^n}{n!}\,(1\cdot 3\cdot 5\cdots(2n-5)\cdot(2n-3))$$

We now introduce some terms to create factorials, multiplying by the appropriate form of 1 (my favorite trick, as all my students know – unless it's adding the appropriate form of zero....):

$$s_n = -\frac{2^n}{n!}\frac{(1)2(3)4\cdots(2(n-2))(2n-3)(2(n-1))(2n-1)2n}{2^{n-2}(n-2)!(2(n-1))(2n-1)2n}$$

That is,

$$s_n = -\frac{2^n}{n!}\frac{(2n)!}{2^n n!(2n-1)}$$

or, in one penultimate frickin' summary,

$$s_n = -\frac{1}{n!}\frac{(2n)!}{n!(2n-1)}$$

Therefore – and I say this with a sense of exhaustion and relief –

$$s_n = -\frac{(2n)!}{n!n!(2n-1)} = -\frac{1}{2n-1}\binom{2n}{n}$$

But, if you can believe it, we're not done. Now we have to solve

$$2xc(x) = 1-\sqrt{1-4x} = 1-\sum_{n=0}^{\infty} s_n x^n = 1-\sum_{n=0}^{\infty} -\frac{1}{2n-1}\binom{2n}{n} x^n$$

for $c(x)$. The good news is that the 0^{th} term of the sum cancels the 1, so that we have

$$2xc(x) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n}{n} x^n$$

and when we divide by $2x$, we obtain

$$c(x) = \sum_{n=1}^{\infty} \frac{1}{4n-2}\binom{2n}{n} x^{n-1}$$

So stepping down the index by 1, we get

$$c(x) = \sum_{n=0}^{\infty} \frac{1}{4(n+1)-2} \begin{pmatrix} 2(n+1) \\ (n+1) \end{pmatrix} x^n.$$

Therefore, we conclude that

$$C_n = \frac{1}{4(n+1)-2} \begin{pmatrix} 2(n+1) \\ (n+1) \end{pmatrix}$$

It seems like the moment when one should celebrate with a cold, refreshing adult beverage. Sadly, however, that's not the answer I was supposed to get! Wikipedia tells me that the Catalan numbers are given by

$$C_n = \frac{1}{n+1} \begin{pmatrix} 2n \\ n \end{pmatrix}$$

But you know what? One of the things I love the most about mathematics is that there's always more than one way to do things, and the fact of the matter is that these expressions are equal!

To see that, let's start with my uglier expression:

$$C_n = \frac{1}{4(n+1)-2} \begin{pmatrix} 2(n+1) \\ (n+1) \end{pmatrix} = \frac{(2(n+1))!}{(n+1)!(n+1)!(4(n+1-2)}$$

So

$$C_n = \frac{(2(n+1))(2n+1)2n\cdots(n+2)}{(n+1)n!2(2(n+1)-1)} = \frac{(2(n+1))(2n+1)2n\cdots(n+2)}{n!2(n+1)(2n+1)}$$

and finally – finally, really truly –

$$C_n = \frac{2n\cdots(n+2)}{n!} = \frac{2n\cdots(n+2)(n+1)}{n!(n+1)} = \frac{2n\cdots(n+2)(n+1)n!}{n!n!(n+1)} = \frac{1}{n+1} \begin{pmatrix} 2n \\ n \end{pmatrix}$$

Repeat:

$$C_n = \frac{1}{n+1} \begin{pmatrix} 2n \\ n \end{pmatrix}$$

of which the first few values are

$$\{1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots\}$$

Gloria in Excelsis Deo! (If there's one thing for certain, it's that God speaks Fibonacci, and Catalan....)

Amen.