Directions: You must skip one problem – write "skip" prominently on it. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1. (10 pts)

a. For the given function (a quartic, of form

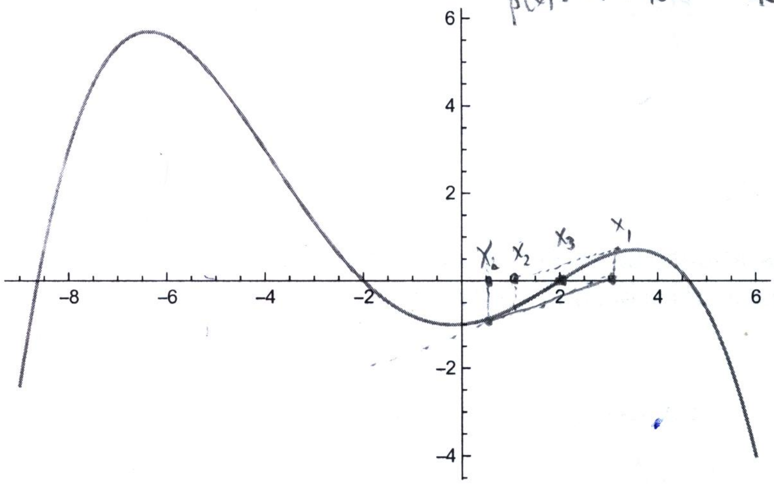
$$p(x) = -1 + x/10 + (11x^2)/40 - x^3/40 - x^4/160$$

we have the following graph:

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = 0.5 - \frac{-0.884766}{0.353125}$

$= 0.5 + 2.505532 = 3.005532$

$p(x) = -1 + \frac{1}{10}x + \frac{11}{40}x^2 - \frac{1}{40}x^3 - \frac{1}{160}x^4$



$x_0 = 0.5$

$x_1 = 3.005532$

a. (4 pts) Use the figure to illustrate how Newton's method works starting from the guess $x = 0.5$. Include the calculation of the next iterate (you don't need to do it in general, just for $x = 0.5$), and you can use these values: $p(0.5) = -0.884766$, and $p'(0.5) = 0.353125$.

b. (2 pts) $p(x)$ is not expressed efficiently for calculation. How might you write it better?

Using Horner's rule which nests the terms.

$$\boxed{p(x) = -1 + x\left(\frac{1}{10} + x\left(\frac{11}{40} + x\left(-\frac{1}{40} + x\left(-\frac{1}{160}\right)\right)\right)\right)}$$

c. (4 pts) The quadratic form of Newton relied on fitting the "tangent quadratic". Knowing that $p''(0.5) = 0.45625$, write the tangent quadratic at $x = 0.5$. (Hint: use Taylor's Theorem.)

$$f(x) \approx p(0.5) + p'(0.5)(x-0.5) + \frac{p''(0.5)(x-0.5)^2}{2}$$

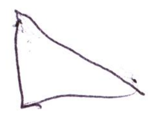
$$\boxed{= -0.884766 + 0.353125(x-0.5) + \frac{0.45625}{2}(x-0.5)^2}$$

Problem 2. (10 pts) We've studied how to compute linear splines and the errors we make in using them. Suppose that we use the "knots" $(x_i, y_i) = (i * h, \sin(x_i))$, where $h = 0.1$ and create a linear spline interpolator for $\sin(x)$. (Calculations below are in radians, of course).

a. (3 pts) Write an explicit formula for the piece of the linear spline $s(x)$ used to estimate $\sin(.55)$. What is the estimate for $\sin(.55)$, and how does it compare to the true value?

$(X_5, Y_5) = (5 \times 0.1, \sin(0.5))$

$(X_6, Y_6) = (6 \times 0.1, \sin(0.6))$



$$S(X) = Y_5 \frac{(X - X_6)}{(X_5 - X_6)} + Y_6 \frac{(X - X_5)}{(X_6 - X_5)}$$

radian = 360 / 2π

57.2957795 1

28.6479

34.3775

Sin (71.5127)

$$\boxed{S(0.55) = \sin(0.5)\left(\frac{0.55 - 0.60}{0.50 - 0.60}\right) + \sin(0.60)\left(\frac{0.55 - 0.50}{0.60 - 0.50}\right)}$$

$$= (0.4794)(0.5) + (0.5646)(0.5) = 0.522$$

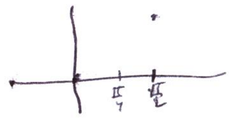
(Quite close)

true answer = 0.5227 ✓

✓

b. (3 pts) We can make some statements about the error we make in estimating $\sin(.55)$ using the linear spline, by considering properties of $\sin(x)$ in the neighborhood of $x = .55$. What can we say?

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$ by taylor approx about $x = 0$



Concavity → linear is an under estimate

$$\sin(x) = \sin(0.55) + \cos(0.55)(x - 0.55) - \frac{\sin(0.55)}{2}(x - 0.55)^2 + \cdots$$

For e^x we get an error bound $\frac{h^2}{8} e'$

For $\sin x$ we get an error bound $\frac{h^2}{8} \cdot 1$

$E(x) = \sin(x) - S(x)$

$E(x)$ continuous, bounded and twice differentiable

c. (3 pts) Bound the error we make in estimating $\sin(.55)$ using the linear spline.

By Thm 5.3

$f'(x) = \cos x$

$f''(x) = -\sin x$

$|f''(x)| \leq 1$

$$|\text{error term}| = \left| (X - X_0)(X - X_1) \frac{f''(\xi)}{2!} \right|$$

can bound by sin(.6)

$$\leq \left| \frac{(0.55 - 0.50)(0.55 - 0.60)}{2} \right|$$

$$= \left| \frac{1}{800} \right| = \boxed{0.00125}$$

We can do better, but it's a bound!

9.5

Problem 3. (10 pts) Consider the three points

x_i	$f[x_i]$		
$x_0 = 0$	1		
		1	
$x_1 = 1$	2		-2.5
		-4	
$x_2 = 2$	-2		

✓

a. (3 pts) Complete the divided-difference table above.

b. (3 pts) Write the Newton interpolating polynomials (from constant to quadratic) obtained by successively adding the points in the order x_1, x_2, and x_0.

$$p(x) = 2 + (x-1)(-4) + (x-1)(x-2)(-2.5)$$

✓

c. (3 pts) Write the interpolating polynomial in Lagrange form.

$$p(x) = (1)\frac{(x-1)(x-2)}{(0-1)(0-2)} + (2)\frac{(x-0)(x-2)}{(1-0)(1-2)} + (-2)\frac{(x-0)(x-1)}{(2-0)(2-1)}$$

✓

d. (1 pt) What is the difference between these polynomials?

There is no difference other than the way they are written.

✓

Well done

Problem 4: (10 pts) Discuss the advantages and disadvantages of each of the following root finding methods. Mention order of convergence when you can, dangers, possibilities, mathematical underpinnings, etc. Show me that you understand each method.

Method	Advantages	Disadvantages
Newton's	• Converges fast (Quadratically)	• Have to calculate 1st derivative • Doesn't converge if the initial guess isn't good enough.
Secant	• Converges at rate of golden ratio • Can use divided differences instead of derivatives.	Need two good initial guesses to converge.
Muller's	Only method (that we've been taught) that finds complex roots from real starting values	• Requires 3 good initial guesses • Must calculate roots of quadratic for every iteration.
Bisection	• Gain 1-bit of accuracy for every iteration • Always converge if IVT applies + zero is trapped	• Doesn't necessarily converge quickly. • Requires a bound around the root, to converge.

Problem 5. (10 pts) We seek a root of

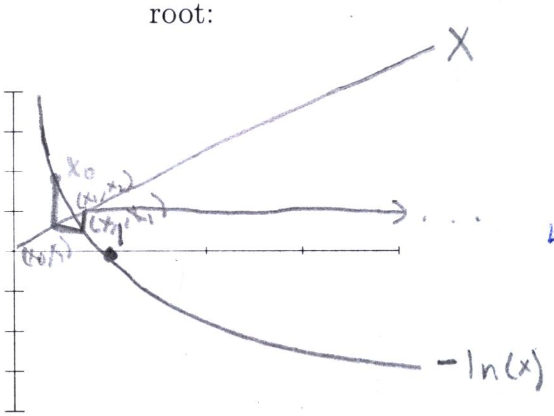
$$f(x) = x + \ln(x)$$

You consider two possibilities, both of them fixed-point methods: Newton's method, and the more straightforward method suggested by

$$f(x) = 0 \iff x = -\ln(x)$$

So define $g(x) = -\ln(x)$.

a. Sketch (roughly) the graphs of x and $-\ln(x)$ below, and conclude that there is a unique root:



The root of f(x) occurs where x and -ln(x) intersect. x is monotonically increasing while -ln(x) is monotonically decreasing, so they must intersect exactly once. Therefore there is a single unique root.

b. Explain why $g(x)$ is a poor choice for a fixed-point iteration scheme.

g(x) = -ln(x) is a poor choice for the straightforward method because it will not "cobweb" to the root.

Why not? $|g'(r)| > 1$

−5

c. Newton's method is also a fixed-point iteration scheme. What is Newton's fixed-point function to find roots of f? $\cancel{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - x f(x_k)}$

Thats it!

write it out...

Problem 5. (10 pts) We seek a root of

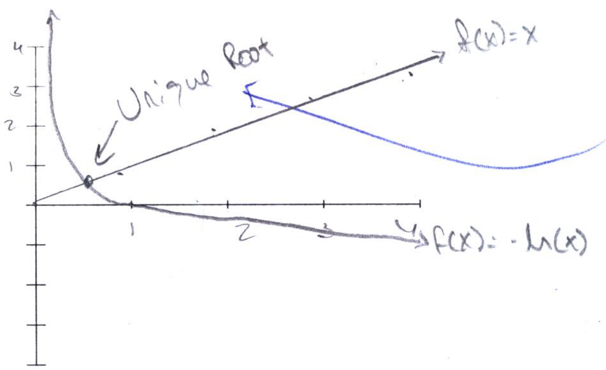
$$f(x) = x + \ln(x)$$

You consider two possibilities, both of them fixed-point methods: Newton's method, and the more straightforward method suggested by

$$f(x) = 0 \iff x = -\ln(x)$$

So define $g(x) = -\ln(x)$.

a. Sketch (roughly) the graphs of x and $-\ln(x)$ below, and conclude that there is a unique root:



Why?

~✓

b. Explain why $g(x)$ is a poor choice for a fixed-point iteration scheme.

g(x) is a poor choice because when we start to the left of the root the slope is >1 (most of the time) + the method will not work. When we start to the right of the root we will end up on the left of the root + the slope will be greater than 1. There is a very small window where this method will work.

very small!! [(it won't)

c. Newton's method is also a fixed-point iteration scheme. What is Newton's fixed-point function to find roots of f?

$$X_{k+1} = X_k - \frac{f(X_k)}{f'(X_k)} \leftarrow \text{in general}$$

for this problem

$$X_{k+1} = X_k - \frac{X_k + \ln(X_k)}{1 + \frac{1}{X_k}}$$

$$X_{k+1} = X_k - \frac{X_k + \ln(X_k)}{X_k + 1} \cdot X_k$$

$$X_{k+1} = X_k\left(1 - \frac{X_k + \ln(X_k)}{X_k + 1}\right)$$

✓

10

Problem 6. (10 pts) Determine a low-degree polynomial approximation to

$$f(x) = \cos(x) - 1$$

for $|x| < 0.1$ having a relative error of less than or equal to 10^{-3} in magnitude. (Hint: Taylor series remainder term, and both numerator and denominator are easy to bound.)

$f'(x) = -\sin(x)$ Taylor Approx

$f''(x) = -\cos(x)$ $= 0 - 0 - \frac{x^2}{2!} - 0 + \frac{x^4}{4!} - 0 + \frac{f^{(6)}(\xi)x^6}{6!}$

$= -\frac{x^2}{2} + \frac{x^4}{4!} +$ remainder

If $x = 0.1 \Rightarrow$ The remainder is $\frac{0.1^6 f^{(6)}(\xi)}{6!}$

$f^{(6)}$ is either a cos or sin term, at worst it could be 1 (or -1) to maximize error

assuming it's one:

Need to set relative error in here.

$\frac{0.1^6 (1)}{6!} \simeq 0.000000001$ which is definitely < 0.001

Try with

$\frac{-x^2}{2}$ +remainder $\frac{0.1^4 (1)}{4!} \simeq 0.00004167$

More work on scratch.

In conclusion, both work but $\frac{-x^2}{2}$ is a lower degree polynomial ← good!

8.5

$$\frac{f(x_0)-f(x_1)}{x_0-x_1} = \frac{1-2}{0-1} = \frac{-1}{-1} = 1$$

$$= \frac{-2-2}{2-1} = \frac{-4}{1} = -4$$

$$\frac{1-(-4)}{2-0} = \frac{5}{2}$$

Problem 6) $f(0.1)$

Check Relative Errors

degree 4 polynomial	degree 2 poly.
Approx = -0.004995833	Approx = -0.005
True = -0.004995835	True = -0.004995835
Rel. Error = -0.0000004 < 0.001 ✓	Rel. Error = -0.00083694 < 0.001 ✓

how does relative get smaller?

0.45