"Showing Joey..."

$$f[x_1,x_2,x_3] = \frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1}$$ (apparent asymmetry in the defn.)

$$= \frac{\dfrac{f(x_3)-f(x_2)}{x_3-x_2} - \dfrac{f(x_2)-f(x_1)}{x_2-x_1}}{x_3-x_1}$$

$$= \frac{(x_2-x_1)(f(x_3)-f(x_2)) - (x_3-x_2)(f(x_2)-f(x_1))}{(x_3-x_1)(x_3-x_2)(x_2-x_1)}$$

$$= \frac{x_3(f(x_1)-f(x_2)) + x_2(f(x_3)-f(x_1)) + x_1(f(x_2)-f(x_3))}{(x_1-x_2) * (x_3-x_1) * (x_2-x_3)}$$

Consider $f[x_2,x_1,x_3] =$

$$= \frac{x_3(f(x_2)-f(x_1)) + x_1(f(x_3)-f(x_2)) + x_2(f(x_1)-f(x_3))}{(x_2-x_1) * (x_3-x_2) * (x_1-x_3)}$$

changed the sign in the denominator! but changed the sign in the numerator too!

$$f[x_1,x_2,x_3] \underline{\underline{=}} f[x_2,x_1,x_3]$$

In fact all permutations of the arguments give the same thing!

Any permutation is made up of pair-wise transpositions:

$x_1 \ x_2 \ x_3$

$x_1 \ x_3 \ x_2$

$x_3 \ x_1 \ x_2$

$x_3 \ x_2 \ x_1$

$x_2 \ x_3 \ x_1$

$x_2 \ x_1 \ x_3$

The Logarythms video!

Theorem 5.6 (Invariance Theorem)

$f[x_0, \dots, x_n]$ is invariant under all permutations!

In numerical differentiation one may be inclined to make h smaller & smaller, hoping for better accuracy: However, we run into problems because of roundoff as we make h <u>too</u> small.

Let's define

$\tilde{f}(x_0) \simeq f(x_0)$ (due to roundoff we compute $\tilde{f}$)

$$\left.\begin{aligned} e(x_0) &= \tilde{f}(x_0) - f(x_0) \\ e(x_0+h) &= \tilde{f}(x_0+h) - f(x_0+h) \end{aligned}\right\} \text{ e is the roundoff error.}$$

Assume ε represents a bound on roundoff error.

$$E \equiv \left| f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} \right| \quad \text{(error in the derivative)}$$

$$= \left| f'(x_0) - \frac{f(x_0+h) - e(x_0+h) - (f(x_0) - e(x_0))}{h} \right|$$

$$= \left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} + \frac{e(x_0+h) - e(x_0)}{h} \right|$$

$$= \left| \frac{h}{2} f''(\xi) + \frac{e(x_0+h) - e(x_0)}{h} \right|$$

$$\leq \left| \frac{h}{2} f''(\xi) \right| + \left| \frac{e(x_0+h) - e(x_0)}{h} \right|$$

$$\leq \underbrace{\frac{h}{2} M + \frac{2\varepsilon}{h}}$$

M: upper bound on $|f''(x)|$; assume $h > 0$

Objective: minimize this bound!

This is a function, we want to minimize it: 1st & 2nd derivatives!

$$B(h) = \frac{hM}{2} + \frac{2\varepsilon}{h}$$

$$B'(h) = \frac{M}{2} - \frac{2\varepsilon}{h^2} = 0$$

demand for an extremum.

$$h^2 = \frac{4\varepsilon}{M} \rightarrow \boxed{h = 2\sqrt{\frac{\varepsilon}{M}}}$$

So now we know how to compute derivatives using Taylor. The linear algebra approach is really nice: find weights w_1 & w_2 /

$$w_1 f(x+h) + w_2 f(x) = f'(x) + \text{error}$$

$$w_1 f(x+h) = w_1 (f(x)) + w_1 h f'(x) + w_1 \frac{h^2}{2} f''(\xi_+)$$

$$+ \quad \underline{w_2 f(x) = w_2 f(x)}$$

$$f'(x) = (w_1 + w_2) f(x) + (w_1 h) f'(x) + w_1 \frac{h^2}{2} f''(\xi)$$

$+$ error

$$\therefore \quad \begin{aligned} w_1 + w_2 &= 0 \\ w_1 h &= 1 \end{aligned} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ h & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{linear system.}}$$

What about higher derivatives?

$$f''(x) \simeq \frac{f'(x+h) - f'(x)}{h}$$ (seems asymmetric...)

$$\simeq \frac{\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h}}{h}$$

$$= \boxed{\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}}$$ symmetry!

Let's get the error of this formula using Taylor series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(iv)}(\xi_+)$$

$$f(x) = f(x)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(iv)}(\xi_-)$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \underbrace{\frac{h^2}{4!}\left(f^{(iv)}(\xi_+) + f^{(iv)}(\xi_-)\right)}_{\frac{h^2}{12} f^{(iv)}(\xi)}$$

by the IVT.

Error $O(h^2)$