

Richardson's Extrapolation

Forward d. difference:

$$\textcircled{1} \quad f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2} f''(\xi_+)$$

$$\textcircled{2} \quad f'(x) \approx \frac{f(x+2h) - f(x)}{2h} + \frac{(2h)}{2} f''(\xi_{2+})$$

Two estimates for the same thing -
how can we combine them to get a
better estimate?

The error of $\textcircled{1}$ is roughly
half the error of $\textcircled{2}$. (depends
on the behavior of f'' - is it
pretty constant on $[x, x+2h]$)

So a good "average" would
be

$$f'(x) \approx 2\textcircled{1} - \textcircled{2}$$

That will cancel the errors (roughly)

$$\begin{aligned}
 f'(x) &\approx 2 \frac{f(x+h) - f(x)}{h} - \frac{f(x+2h) - f(x)}{2h} \\
 &= \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} \\
 &\approx \left[\frac{-\frac{1}{2}f(x+2h) + 2f(x+h) - \frac{3}{2}f(x)}{h} \right]
 \end{aligned}$$

& we can hope that the
 error is $O(h^2)$!