

Runge's Trapezoidal Method

(also takes Taylor-2)

$$y_{n+1} = y_n + \frac{h}{2} \left[f(t_n, y_n) + f(t_n+h, y_n + \underbrace{h f(t_n, y_n)}_k) \right]$$

$$= y_n + \frac{h}{2} \left[f(t_n, y_n) + f(t_n, y_n) + \frac{\partial f}{\partial t} \cdot h + \frac{\partial f}{\partial y} h f(t_n, y_n) + O(h^2) \right]$$

$$= y_n + h f(t_n, y_n) + \frac{h^2}{2} \left(\frac{\partial f}{\partial t} + f(t_n, y_n) \frac{\partial f}{\partial y} \right) \Big|_{t_n, y_n} + O(h^3)$$

$$= \text{Taylor-2} + O(h^3)$$

Euler 1.5

$t_n = 2$

Taylor 2 1.57

$y_n = 1$

Runge Mid 1.5765

$h = 0.1$

Trap 1.583

$y'(t) = t^2 + y(t)^2$

RK-4

$= f(t, y(t))$

Euler: $y_n + h f(t_n, y_n)$

$= 1 + 0.1 * (2^2 + 1^2) = 1.5$

Taylor-2: $y_n + h \left[f(t_n, y_n) + \frac{h}{2} \left(\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \right) \right] \Big|_{t_n, y_n}$

$= 1 + 0.1 * \left[5 + \frac{0.1}{2} (2 \cdot 2 + 5 \cdot 2 \cdot 1) \right]$

$= 1 + 0.1 [5 + .05 \cdot 14]$

$= 1.57$

RK-mid: $y_n + h \left[f \left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n) \right) \right]$

$= 1 + 0.1 \left[f(2.05, 1 + .05 \cdot 5) \right]$

$= 1 + 0.1 \left[f(2.05, 1.25) \right]$

$= 1 + 0.1 (2.05^2 + 1.25^2) = 1.5765$