Runge Kutta Methods

approximate Taylor methods (but without the derivatives).

Runge's Midpoint Method

approximates Taylor-2

$$g_1 = f(t_n, y_n)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2} g_1$$

$$y_{n+1} = y_n + h g_2 \quad \text{where}$$

$$g_2 = f\left(t_n + \frac{h}{2}, y_{n+\frac{1}{2}}\right).$$

$$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$$

Now we expand this about (t_n, y_n):

$$f\left(t_n+\frac{h}{2},\, y_n+\frac{h}{2} f(t_n, y_n)\right) =$$

$$f(t_n, y_n) + \frac{h}{2} \frac{\partial f}{\partial t}\bigg|_{(t_n, y_n)}$$

$$+ \frac{h}{2} f(t_n, y_n) \frac{\partial f}{\partial y}\bigg|_{(t_n, y_n)} + O(h^2)$$

$$y_{n+1} = y_n + h\left[f(t_n, y_n) + \frac{h}{2}\left(\frac{\partial f}{\partial t} + f(t_n, y_n)\frac{\partial f}{\partial y}\right) + O(h^2) \right.$$

$$= y_n + h f(t_n, y_n) + \underbrace{\frac{h^2}{2}\left(\frac{\partial f}{\partial t} + f(t_n, y_n)\frac{\partial f}{\partial y}\right)} + O(h^3)$$

~~Replaces~~ Matches the

$O(h^2)$ term

in Taylor-2

will be different

from the error term

in Taylor-2