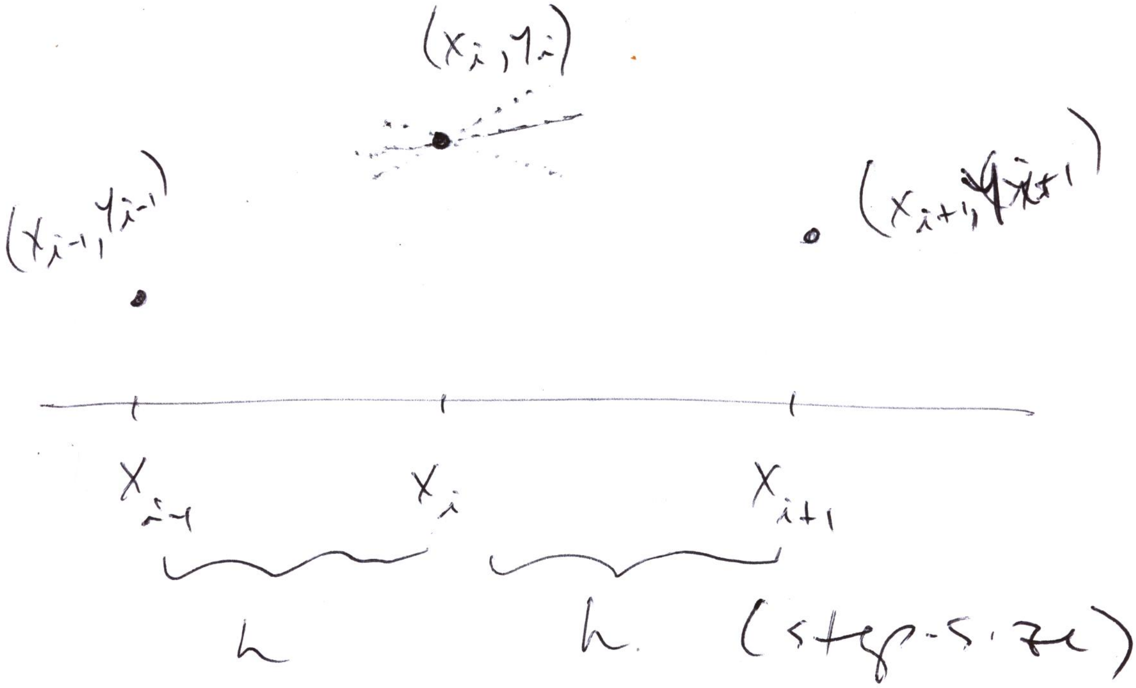
Differentiation (Numerical!)



$y_i = f(x_i)$ (often, but not always!)
(may just be data)

Forward difference: $f'(x_i) \simeq \dfrac{f(x_{i+1}) - f(x_i)}{h}$

$\left(= f[x_{i+1}, x_i] \right)$

Backward: $f'(x_i) \simeq \dfrac{f(x_i) - f(x_{i-1})}{h}$.

<u>Centered</u> (the average!): $f'(x_i) \simeq \dfrac{f(x_{i+1}) - f(x_{i-1})}{2h}$

For centered I want to use one more term from Taylor:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(\xi_+)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f'''(\xi_-)$$

<u>Centered</u>: $$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^3}{3!}\left(\frac{f'''(\xi_+) + f'''(\xi_-)}{2h}\right)$$

Error: $$\left| f'(x) - f[x+h, x-h] \right| = O(h^2)$$

Goes to 0 like <u>h^2</u> - much better! Provided we have boundedness of $f'''(x)$ on the interval $[x-h, x+h]$

Why restrict ourselves to only 3 points? Why not 5, centered? Why not 4 forward!?

What errors are we making?
(turn to Taylor)

Consider

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi_+)$$

$$f(x) = f(x)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(\xi_-)$$

---

forward: $\dfrac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2} f''(\xi_+)$

backward: $\dfrac{f(x) - f(x-h)}{h} = f'(x) - \frac{h}{2} f''(\xi_-)$

---

errors:

$$|f'(x) - f[x+h, x]| = O(h)$$

$$|f'(x) - f[x, x-h]| = O(h)$$

order of accuracy

as $h \to 0$, the error goes to zero (as long as f'' is bounded)

Linear algebra can help us derive these more complicated schemes. Let's re-derive the forward difference formula.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi_+)$$

$$f(x) = f(x)$$

We want a weighted combination that gives us $f'(x)$ (+ some error).

$$w_1 f(x) + w_2 f(x+h) = 0 \cdot f(x) + 1 \cdot f'(x)$$

(+ error)

Using the Taylor series

$$w_1 \cdot f(x) + w_2 f(x) = 0 \cdot f(x)$$

$$w_1 \cdot 0 + w_2 \cdot hf'(x) = 1 \cdot f'(x)$$

This is a system of linear equations

$$\begin{bmatrix} 1 & 1 \\ 0 & h \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Bigg\}$$

solution:

$w_2 = 1/h$

$w_1 = -1/h$

So the combination

$$-\frac{1}{h} \cdot f(x) + \frac{1}{h} f(x+h) =$$

$$\frac{f(x+h) - f(x)}{h}$$

(forward difference!)