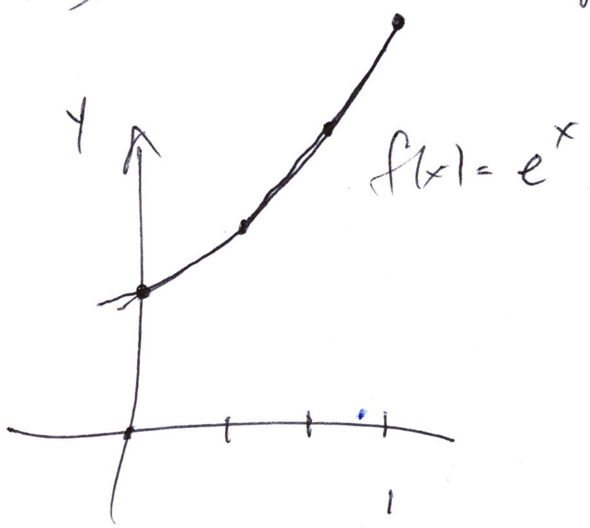
§5.5 – Piecewise Polynomial Interpolation (splines)

Compute e^x on $[0,1]$ using linear interpolants (C^0)

continuity (no slope continuity)

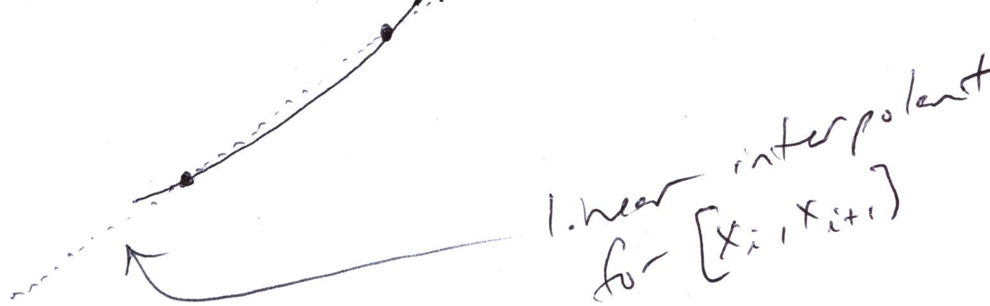


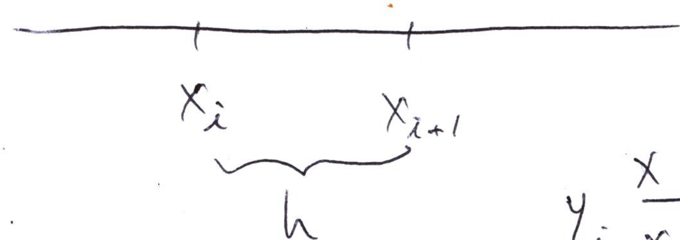
How many points do we need to keep the error under 10^{-6}?

$$|e^x - s(x)| < 10^{-6} \text{ for } x \in [0,1].$$

where s is the linear spline.

We'll use equal spacing for the points, of $h = \frac{1-0}{N} = \frac{1}{N}$ where N is the number of points.





$$y_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + y_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \equiv l_i(x)$$

Define $E(x) = e^x - s(x)$

$E(x_i) = 0$ ($s(x)$ interpolates e^x at x_i !)

Between x_i + x_{i+1} both e^x and $s(x)$ are twice differentiable, so $E(x)$ is also twice differentiable.

$E(x_i) = E(x_{i+1}) = 0$. By the Extreme Value Theorem, there have to be extreme values(!) between x_i + x_{i+1}. So $|E(x)|$ will achieve a max on (x_i, x_{i+1}) without loss of generality (3)

Define x_m as the place where the max occurs. WLOG we can assume x_m is closer to x_i:

$$x_m = x_i + \theta h \quad \text{where } \theta \in [0, \tfrac{1}{2}]$$

Since x_m is associated with a maximum,

$$E'(x_m) = 0 \quad (\text{Fermat's Theorem})$$

So let's do a Taylor series expansion of $E(x_i)$ about x_m:

$$\underbrace{E(x_i)}_{=0!} = E(x_m) + \underbrace{E'(x_m)}_{=0}(-\theta h) + E''(\xi)\frac{(-\theta h)^2}{2!}$$

interpolator! This exists!

$$0 = E(x_m) + E''(\xi)\frac{(-\theta h)^2}{2!}$$

So $|E(x)| \le |E(x_m)| \le \left| E''(\xi)\frac{(-\theta h)^2}{2} \right|$

on $[x_i, x_{i+1}]$

$$E''(x) = \left(e^x - g(x)\right)'' = e^x \quad \text{on } (x_i, x_{i+1})$$

What's the worst that

$$\left| \frac{E''(\xi)(-\theta h)^2}{2} \right|$$

could be on $x \in [0,1]$?

$$\left| \frac{E''(\xi)(-\theta h)^2}{2} \right| \leq \frac{e \cdot h^2}{8} \quad (p198)$$

We demand $\frac{e h^2}{8} < 10^{-6}$

$$h < \sqrt{\frac{8}{e} \cdot 10^{-6}} \quad \left(\text{gives } N = 583\right)$$

$h = .001$ will work - so a thousand points would be excessive ...

Given $l_i(x)$ for $i \in \{0, \dots, N-1\}$

How do we write $s(x)$, the linear <u>spline</u> for $x \in [0,1]$?

Given x, what <u>i</u> do we choose (gives $l_i(x)$)?

$i = \lfloor x \cdot N \rfloor$ $\qquad \lfloor\ \rfloor \equiv$ floor function

$x = 0$: $i = 0$

$x = 1$: $\lfloor x \cdot N \rfloor = \lfloor N \rfloor = N$ and we don't have a function l_N. So we have a slight problem, but that's the basic idea.

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s[x_] := l[Floor[x*N]][x]
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Let's fit a cubic to two points

(x_0, y_0) & (x_1, y_1)

with given slopes $f'(x_0) = m_0$ &

$f'(x_1) = m_1$.

Let's start with Lagrange's linear interpolant to the two points,

$$\ell(x) = y_0 \frac{(x - x_1)}{(x_0 - x_1)} + y_1 \frac{(x - x_0)}{(x_1 - x_0)} \qquad \left(\ell'(x) = m \text{ where } m = \frac{y_1 - y_0}{x_1 - x_0} \right)$$

So now need to add some cubic terms that "stay out of the way" at x_0 & x_1, except to handle the slopes!

$$c_0(x) = (x - x_0)(x - x_1)^2 a \qquad \left(\begin{matrix} \text{handle at} \\ x_0 \end{matrix} \right)$$

$$c_1(x) = (x - x_0)^2 (x - x_1) b$$

$$S'(x_0) = m_0 = \ell'(x_0) + c_0'(x_0) \,.$$

$$S'(x_1) = m_1 = \ell'(x_1) + c_1'(x_1)$$

So $\quad c_0'(x_0) = m_0 - m$

$$c_1'(x_1) = m_1 - m$$

So $\boxed{s(x) = l(x) + c_0(x) + c_1(x)}$

$$c_0'(x)\Big|_{x=x_0} = (x-x_1)^2 a\Big|_{x=x_0} = (x_0-x_1)^2 a = m_0 - m$$

$$\therefore \quad a = \frac{m_0 - m}{(x_0 - x_1)^2}$$

& by symmetry

$$b = \frac{m_1 - m}{(x_1 - x_0)^2} \left(= \frac{m_1 - m}{(x_0 - x_1)^2} \right)$$