Euler's error & step-size control

Theorem: Suppose f is cont. & satisfies a Lipschitz condition w/ constant L on

$$D = \{(t,y) \mid a \le t \le b,\ -\infty < y < \infty\}$$

& $\exists M > 0$ / $|y''(t)| \le M \quad \forall t \in [a,b]$.

Let $y(t)$ denote the unique soln to

$$y'(t) = f(t, y(t)) \qquad a \le t \le b$$
$$y(a) = \alpha$$

& $w_0, w_1, \dots, w_N$ be the Euler approximations, with $h = \frac{b-a}{N}$.

Then for each $i \in \{0, \dots, N\}$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

<u>Proof</u>: When $i=0$ the result holds

since $w_0 = y(t_0) = y(a) = \alpha$

So $|y(t_0) - w_0| = 0$!

Since

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2} y''(\xi_i)$$

Euler says: replace
$y'(t_i)$ with $f(t_i, y(t_i))$
& throws away the
$y''(\xi_i)$ part -
Local truncation error

& that becomes

$$w_{i+1} = w_i + h f(t_i, w_i) \quad \left(\begin{array}{c}\text{Euler}\\ \text{iterates}\end{array}\right)$$

So the error

$$y_{i+1} - w_{i+1} = y_i - w_i + h\left[f(t_i, y_i) - f(t_i, w_i)\right] + \frac{h^2}{2} y''(\xi_i)$$

By the triangle inequality

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i| + \left|h\left(f(t_i, y_i) - f(t_i, w_i)\right)\right| + \left|\frac{h^2}{2} y''(\xi_i)\right|$$

& using the Lipschitz condition,

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i| + hL|y_i - w_i| + \frac{h^2}{2}|y''(\xi_i)|$$

$$(*) \qquad \le (1+hL)|y_i - w_i| + \frac{h^2}{2}M$$

Lemma: If s & t are positive

a_i satisfies $a_0 \ge -t/s$ and

$a_{i+1} \le (1+s)a_i + t \quad \forall i \ge 0,$

then $a_{i+1} \le e^{(i+1)s}(a_0 + t/s) - t/s$

Proof: $a_{i+1} \le (1+s)a_i + t$

$$\le (1+s)\left[(1+s)a_{i-1} + t\right] + t$$

$$\le \cdots$$

$$\le (1+s)^{i+1}a_0 + \underbrace{\left[1 + (1+s) + \cdots + (1+s)^i\right]}_{\frac{1-(1+s)^{i+1}}{1-(1+s)}} t$$

so

$$a_{i+1} \le (1+s)^{i+1}a_0 + \left[\frac{1}{s}\left[(1+s)^{i+1} - 1\right]\right] t$$

$$a_{i+1} \leq (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - t/s$$

Since

$$1 + x \leq e^x$$

$$(1+x)^{i+1} \leq \left(e^x\right)^{i+1} = e^{x(i+1)}$$

$$a_{i+1} \leq e^{s(i+1)}\left(a_0 + t/s\right) - t/s$$

QED

In $(*)$ a_{i+1} is playing the role of $|y_{i+1} - w_i|$; s is playing hL, + t is playing $\frac{h^2M}{2}$.

$$\therefore \quad |y_{i+1} - w_{i+1}| \leq e^{hL(i+1)}\left(\underbrace{|y_0 - w_0|}_{=0} + \frac{h^2M/2}{hL}\right) - \frac{hM}{2L}$$

So $|y_{i+1} - w_{i+1}| \leq e^{hL(i+1)}\left(\frac{hM}{2L}\right) - \frac{hM}{2L}$

$$= \frac{hM}{2L}\left(e^{hL(i+1)} - 1\right)$$

Or

$$|y_i - w_i| \le \frac{hM}{2L}\left(e^{hLi} - 1\right)$$

$$= \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right)$$

Q.E.D.

More General Taylor Methods.

Euler is Taylor-1
(we use the tangent line - linear)

Why not use the "tangent quadratic"?

$$U(t_{i+1}) \simeq U(t_i) + h\underbrace{U'(t_i)}_{f(t_i,\, U(t_i))} \quad \text{(Euler)}$$

Consider

$$U(t_{i+1}) \simeq U(t_i) + h\underbrace{U'(t_i)}_{f(t_i,\, U(t_i))} + \frac{h^2}{2}\underbrace{U''(t_i)}_{\text{What about this?}}$$

$$U'(t) = f(t, U(t))$$

$$u''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{du}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} f(t, u(t))$$

For example - p340

$$u'(t) = t^2 + u(t)^2 \quad (\equiv f(t, u(t)))$$

$$\frac{\partial f}{\partial t} = 2t.$$

$$\frac{\partial f}{\partial u} = 2u$$

$$u''(t) = 2t + 2u(t)\left(t^2 + u(t)^2\right)$$

$$u(t_{i+1}) \simeq u(t_i) + h\left(t_i^2 + u(t_i)^2\right) +$$
$$\frac{h^2}{2}\left(2t_i + 2u(t_i)\left(t_i^2 + u(t_i)^2\right)\right)$$

Taylor-2