Earth's Orbit

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The Milankovitch cycles are primary determinants of Earth's climate:

"Changes in the shape of the Earth's orbit (or eccentricity) as well as the Earth's tilt and precession affect the amount of sunlight received on the Earth's surface. These orbital processes – which function in cycles of 100,000 (eccentricity), 41,000 (tilt), and 19,000 to 23,000 (precession) years – are thought to be the most significant drivers of ice ages according to the theory of Mulitin Milankovitch, a Serbian mathematician (1879-1958). The National Aeronautics and Space Administration's (NASA) Earth Observatory offers additional information about orbital variations and the Milankovitch Theory." (source)

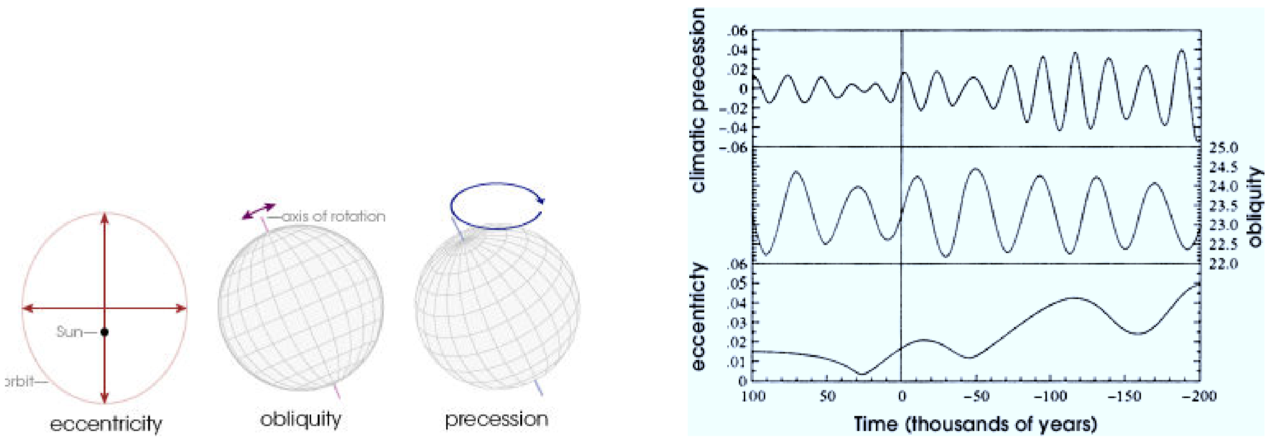


Figure 1: Sources: Left Figure: NASA Right Figure: NASA

In this lab we focus on the eccentricity in Earth's orbit. The fourth page of this lab is a fact sheet, derived from one of the links below.

Links:

- The Milankovitch cycles
- The Motion of Planets (from which much of this lab is derived)
- Earth Fact Sheet
- The Restless Earth

Question 1: Parametric equations for a planetary orbit

The sun is at the origin and the plane of the orbit has coordinates x and y. We can write parametric equations for the time t, and coordinates x and y, in terms of an independent variable ψ:

$$t = \frac{T}{2\pi}\left(\psi - \varepsilon \sin\psi\right) \tag{1}$$

$$x = a\left(\cos\psi - \varepsilon\right) \tag{2}$$

$$y = a\sqrt{1-\varepsilon^2}\sin\psi \tag{3}$$

The fixed parameters T, a, and ε are

- T = period of revolution
- a = semimajor axis
- ε = eccentricity

1. The eccentricity of the Earth's orbit is currently about 0.01671022. Its semimajor axis is 1.00000011 AU.

   Make a parametric plot of the orbit of Earth. (Note: You only need the parametric equations for x and y, letting the variable ψ go from 0 to 2π for one revolution.)

2. Calculate the perihelion distance (distance between the sun – which is at one focus – and the closer point along the semimajor axis. In the figure on the fact sheet, this distance would be 0.5). Express the result in AU.

3. Calculate the aphelion distance (distance between the sun – which is at one focus – and the farther point along the semimajor axis. In the figure on the fact sheet, this distance would be 1.5). Express the result in AU.

4. Compare the orbit of the Earth, given its current eccentricity, with the orbit obtained with an eccentricity of $\varepsilon = 0.5$.

5. Compute the speed of the Earth as a function of time t:

   $$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)}$$

   You will need to do this implicitly, since we are given $t(\psi)$, rather than $\psi(t)$.

   Answer: $2\pi\sqrt{\dfrac{1+\varepsilon\cos\psi}{1-\varepsilon\cos\psi}}$

Question 2: The Newtonian theory of motion and gravity is very accurate, but not exact. A more precise theory was developed by Albert Einstein. It is called the theory of general relativity. In Einstein's theory the orbit of a planet must be a precessing ellipse rather than a perfect closed ellipse. Einstein's theory agrees with planetary observations.

A mathematical representation of a precessing ellipse may be given by the parametric equations

$$x_p = x\cos(0.02\psi) - y\sin(0.02\psi) \tag{4}$$

$$y_p = x\sin(0.02\psi) + y\cos(0.02\psi) \tag{5}$$

where x_p and y_p are the points on the precessing ellipse, and x and y are the points on the nonprecessing ellipse (given by equations (2) and (3) above); ψ is the same parameter as in (2) and (3).

Create a plot of the precessing ellipse if the semimajor axis is a = 1 and the eccentricity is $\varepsilon = 0.5$. Let ψ run for several periods, so that you will actually see the precession of the ellipse.

Mathematics of Motion III
THE MOTION OF PLANETS

History of science

Copernicus ... The planets revolve around the sun.

Galileo ... Observations by telescope prove that the Copernican theory is correct.

Kepler ... Three laws of orbits:
1. The orbits are ellipses with the sun at one focus.
2. The radial vector sweeps out equal areas in equal times.
3. $T^2 \propto a^3$.

Newton ... There are *mathematical theories* that explain why the planets move as they do.
(laws of motion, universal gravity, calculus)

Table of variables

R	radius for a circular orbit
a	semimajor axis of an ellipse
T	period of revolution
m	mass of a planet or satellite
M	mass of the sun
G	Newton's gravitational constant

The period of revolution

This calculation is for the case of a circular orbit:

• Newton's second law says $F = ma$.

• The centripetal acceleration in $a = v^2/R$ where R is the radius of the orbit and v is the speed ($=2\pi R/T$).

• The gravitational force is $F = GMm/R^2$.

• Combine these equations $\Longrightarrow$

$$\frac{m(2\pi R/T)^2}{R} = \frac{GMm}{R^2};$$

and solve for the period of revolution T,

$$T = \sqrt{\frac{4\pi^2 R^3}{GM}}.$$

For the case of an elliptical orbit,

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}}$$

where a is the semimajor axis.

Ellipse geometry

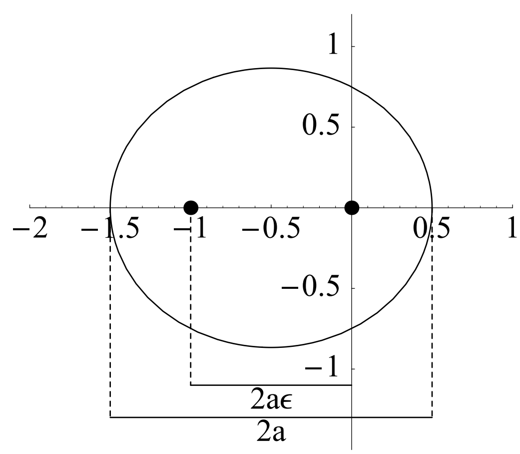
An ellipse may be specified by two fixed parameters,

$$\begin{aligned} a &= \text{semimajor axis,} \\ \varepsilon &= \text{eccentricity.} \end{aligned}$$

The large diameter is $2a$.

The distance between the foci is $2a\varepsilon$.

Note that ε is dimensionless, and must be between 0 and 1. An ellipse with $\varepsilon = 0$ is a circle.



Units

1 AU = 1 astronomical unit = mean distance from sun to Earth = 1.496×10^{11} m.

1 y = 1 year = period of revolution of Earth = 3.15×10^7 s.