2021 John O'Bryan Mathematics Competition
5-Person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper.** Questions will not be scored without the following two things:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. Teams must **show complete solutions (not just answers)** to receive credit. More specific instructions are read verbally at the beginning of the test.

1. (a) Find prime numbers $p < q < r < s$ satisfying $p^q \cdot r \cdot s = 12008$ and $q \cdot r^p = 1083$

   (b) Consider 120308, 1203308, 12033308, and 120333308. Which are divisible by r from the previous part?

   (c) Is $120\underbrace{33\ldots3}_{2021}08$ divisible by r from the first part of this question?

2. A *ribbon of length* k is a rectangle with dimensions $1 \times k$ for some natural number k.

   (a) Consider covering areas with ribbons of distinct lengths; that is, every ribbon used to cover the area has a different length. What is the maximum area that can be covered if the longest ribbon has length $k = 5$? $k = 10$? $k = 15$? $k = 20$? general k?

   (b) If the longest ribbon has length 15, what is the maximum value of m such that an $m \times 15$ rectangle can be covered using ribbons of distinct lengths?

   (c) Assume $m \geq n$. How large must m be so that an $m \times n$ rectangle can be covered using ribbons of distinct lengths?

3. (a) For real numbers x and y, show that $|x + y| \leq |x| + |y|$. (Hint: if $a, b \geq 0$ with $a^2 \leq b^2$, then $a \leq b$.)

   (b) For real numbers x, y, and z, show that $|x+y+z| \leq |x+y-z|+|x-y+z|+|-x+y+z|$.

4. Consider coloring the edges of a prism using 3 colors, say red (r), blue (b), and green (g). A face of the prism "sees" a color if (at least) one of the edges along the border of the face is colored with that color. A vertex of the prism "sees" as color if (at least) one of the three edges touching the vertex is colored with that color.

   (a) Can a triangular prism be colored so that every face and every vertex see all 3 colors?

   (b) Can a hexagonal prism be colored so that every face and vertex see all 3 colors?

   (c) Can a rectangular prism be colored so that every face and vertex see all 3 colors?

5. The integers from 1 to n are written in a row. The same numbers are written under them in a different order (see below). Note that there are no repeated numbers in a row.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \hline 4 & 2 & 1 & 5 & 3 \end{array}$$

Notice that the sum of numbers in the middle three columns is a perfect square. Is it possible that the sums of columns are all perfect squares when

(a) $n = 9$?

(b) $n = 11$?

(c) $n = 1992$?

6. The side BC of an equilateral triangle ABC is divided into three equal parts by the points K and L with K closer to B. The point M divides the side AC in the ratio $AM : MC = 1 : 2$.

(a) Prove ΔMKC is equilateral.

(b) Prove $\overline{AB} \parallel \overline{MK}$.

(c) Prove $\angle AKM = \angle CAL$.

(d) Prove $\angle AKM + \angle ALM = 30°$.

2021 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. Determine the sum of the smallest and largest solutions for the inequality $|2x-5| \le 11$.

2. On the first quiz of a new unit, Terri worked 4 out of 5 problems correctly. On the second quiz she was able to answer twice as many questions correctly, but earned only half as good a grade. Determine the number of questions on the second quiz (there is no partial credit involved in the scoring).

3. Six monkeys all eating at the same rate r can eat 8 bananas in 4 minutes. How many minutes will it take 8 monkeys, eating at the same rate r, to eat 12 bananas? Give your answer as an exact decimal.

4. Determine <u>the sum</u> of all positive values of k such that $x^2 + kx + 24$ can be factored into two binomial factors with integer coefficients.

5. Points $A(-3,8)$ and $B(5,2)$ lie in a plane. Point C lies in quadrant one of the same plane, on the perpendicular bisector of $\overline{AB}$, and 15 units from the midpoint of $\overline{AB}$. Determine the coordinates of point C. Express your answer as an ordered pair (x, y).

6. There are 30 students on a school bus. 24 either play in the band or sing in chorus (or both). 6 play in band but do not sing in chorus. 14 sing in chorus and play in the band. A student is selected at random. Determine the probability that the student sings in the chorus but does not play in band. Express your answer as a common fraction reduced to lowest terms.

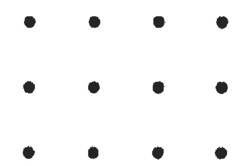
7. Express the number $5.3\overline{25}$, where the "25" is the repeating block of digits, as an improper fraction reduced to lowest terms.

8. One of the angles of an isosceles trapezoid has a degree measure of 42. Determine the largest possible sum of the degree measures of two of the other angles of the trapezoid.

9. The rational expression $\dfrac{\left(3-\dfrac{2}{1-x}\right)}{\left(\dfrac{3}{x-1}-1\right)} = \dfrac{kx+w}{p+qx}$ where k, w, p, and q are integers with $k > 0$. Determine the sum $(k+w+p+q)$.

10. $\overline{AC}$ is the diameter of Circle O. Point B lies on the circle such that $\angle BAC = 60°$ and $AB = 10$. Determine the exact length of the radius of Circle O.

11. An array of 12 points is set up in 3 evenly spaced rows and 4 evenly spaced columns (forming 6 adjacent congruent squares if connected horizontally and vertically). One point is chosen at random from each row. Determine the probability that the three points chosen will be collinear. Express your answer as a common fraction reduced to lowest terms.



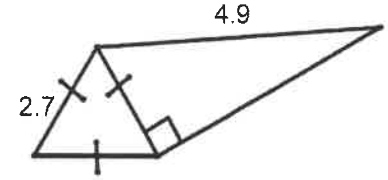
12. A palindromic number is a positive integer that reads the same backwards and forwards, such as 83238. Jayden is challenged to produce the largest palindromic number possible, subject to the requirements that no digit is 0, at least four different digits are used, and the sum of all digits in the number is 25. Jayden is successful in his offering of N. Determine the sum of the first five digits of N (when reading the digits of N from left to right.)

13. The total cost of prom this year was \$17,850. The cost was to be shared equally by those who planned to attend. Regrettably, at the last minute, 10 people could not attend, raising the cost for each person by \$4.25. Determine the number of students who actually attended prom.

14. The product of two positive numbers is 768 and their sum is 7 times their positive difference. Determine the larger of the two numbers.

15. List all real numbers x such that $\left(3x^2-2x\right)^{6x}=1$.

16. A quadrilateral is formed by an equilateral triangle sharing a side with one leg of a right triangle. A side of the equilateral triangle measures 2.7 units, while the hypotenuse of the right triangle measures 4.9 units. Determine the area of this quadrilateral. Express your answer as a decimal rounded to 4 significant digits.



17. The area of a rectangle is 24 square units. The perimeter of the rectangle is 20 units. Find the exact length of the diagonal of the rectangle. Give your answer in the form $a\sqrt{b}$ where b is as small as possible.

18. Three neighbors have identical adjacent back yards. It takes Andrea 60 minutes to mow his yard alone; it takes Bram 54 minutes to mow his yard alone; it takes Claire 36 minutes to mow her yard alone. One week, they start at the same time and work together to mow all three yards. All mow until all three yards are finished. Determine the time <u>in hours</u> they mow together. Express your answer as a common or improper fraction reduced to lowest terms.

19. In the base ten alphanumeric equation $\begin{array}{r} B\ C \\ \times \quad D \\ \hline B\ B\ B \end{array}$, each letter represents a unique digit. Determine the value of the digit B.

20. Trailing zeros are the zeros to the right of the last non-zero digit in the integer representation of a number read from left to right. How many trailing zeros does the number $n = 2021!$ have?

Name: ______________________ Team Code: _______

2021 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________________ 11. ______________________

2. ______________________ 12. ______________________

3. ______________________ 13. ______________________

4. ______________________ 14. ______________________

5. ______________________ 15. ______________________

6. ______________________ 16. ______________________

7. ______________________ 17. ______________________

8. ______________________ 18. ______________________

9. ______________________ 19. ______________________

10. ______________________ 20. ______________________

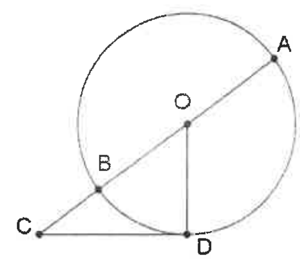
2021 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. Let $f(x) = x^3$. Determine the value of $f^{-1}(8)$.

2. The area of a rectangle is 24 square units. The perimeter of the rectangle is 20 units. Find the exact length of the diagonal of the rectangle. Give your answer in the form $a\sqrt{b}$ where b is as small as possible.

3. In the diagram on the right (not drawn to scale), points A, B, and D lie on circle O, $\overline{CD}$ is tangent at point D, and A, B, and C are collinear. $\overline{AB}$ is a diameter, $CD = 12$, and $CB = 4$. Determine the area of ΔCOD.



4. Let $\left|\frac{x-3}{1-x}\right| \leq x - 1$. Determine the least value of x such that this inequality is true.

5. In a geometric sequence, the last term is 1458, the common ratio is -3, and the sum of the terms is 1094. Determine the second term of this geometric sequence.

6. Alix rolls two fair, standard cubical dice. Determine the probability the sum of the numbers on the upper faces will be equal to or greater than 5. Give your answer as a fraction reduced to lowest terms.

7. Let $\log_b 5 = x$ and $\log_b 3 = y$. Then $\log_b \sqrt[3]{135b} = kx + wy + p$ where k, w, and p are real numbers. Determine the value of the sum $(k + w + p)$. Give your answer as an improper fraction reduced to lowest terms.

8. Let $p(x) = x^2$, $q(x) = 2x + 4$, $f(x) = (p \circ q)(x)$, and $g(x) = (q \circ p)(x)$. Determine the value of x such that $f(x) = 2g(x)$. Give your answer as a fraction reduced to lowest terms.

9. Let $x = \sqrt{33 + \sqrt{33 + \sqrt{33 + \cdots}}}$. Then $x = \frac{k+p\sqrt{w}}{q}$ in simplified and reduced radical form where k, w, p, and q are positive integers. Determine the value of the sum $(k + w + p + q)$.

10. Determine the value of k such that $\frac{2}{6^{x+1}} + \frac{3}{6^x} = \frac{5(2^k)}{6^{x+1}}$ for all real values of x.

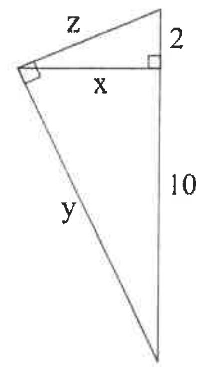
11. Angle A lies in quadrant IV with $\tan(A) = -\frac{3}{4}$. Determine the value of $\sin(2A)$. Give your answer as a proper or improper fraction reduced to lowest terms.

12. Four preschoolers randomly toss a total of 24 indistinguishable balls into three boxes labeled A, B, C. Each ball is equally likely to end up in any one of the boxes and all balls go into one of the boxes. Determine the number of distinct outcomes that are possible. The *order* in which the balls are placed in the containers does not matter, just the outcome.

13. Let $f(x) = \sqrt{a}\,x^2 + \sqrt{ab}\,x + b\sqrt{a}$ with $a > 0$ and $b > 0$. Determine the minimum value for $f(x)$ in terms of a and b

14. Trailing zeros are the zeros to the right of the last non-zero digit in the integer representation of a number read from left to right. How many trailing zeros does the number $n = 2021!$ have?

15. The triangle shown on the right, $(x + y + z) = 2\left[\sqrt{k} + \sqrt{w} + \sqrt{p}\right]$ in simplified and reduced radical form with k, w, and p positive integers. Determine the sum $(k + w + p)$.



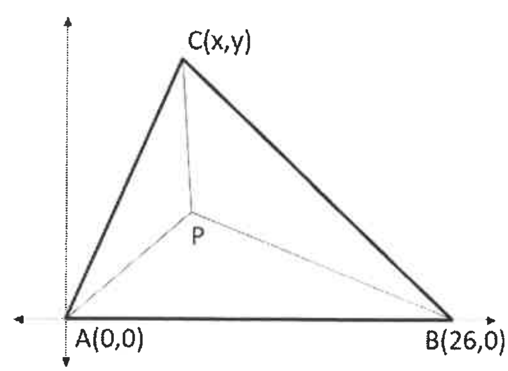
16. An array of 12 points is set up in 3 evenly spaced rows and 4 evenly spaced columns (forming 6 adjacent congruent squares if connected horizontally and vertically). One point is chosen at random from each row. Determine the probability the 3 points chosen will be collinear. Give your answer as a fraction reduced to lowest terms.

17. A triangle has sides with lengths 5, 6, and 8. The area of this triangle can be expressed as $\frac{k\sqrt{w}}{f}$ in simplified and reduced radical form where k, w, and f are positive integers. Determine the value of $(k + w + f)$.

18. Determine the value for x such that $\log_5(5\log_5(\log_5(x^{-5}))) = 1$. Give your answer as a proper or improper fraction reduced to lowest terms.

19. Determine the area of ΔABC when $\angle A = 55°$, $AC = 12$, and $AB = 18$. Express your answer as a decimal rounded to four significant digits.

20. In the diagram of ΔABC on right with coordinates as shown, $AC = 51$, and $BC = 55$. The ratio of the area of ΔPAB to the area of ΔPAC to the area of ΔPBC is $1:2:3$. Determine the **ordered pair** of coordinates that represents point P. Each coordinate should be written as an improper fraction reduced to lowest terms.



Name: ______________________ Team Code: _______

2021 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________	**11.** ______________
2. ______________	**12.** ______________
3. ______________	**13.** ______________
4. ______________	**14.** ______________
5. ______________	**15.** ______________
6. ______________	**16.** ______________
7. ______________	**17.** ______________
8. ______________	**18.** ______________
9. ______________	**19.** ______________
10. ______________	**20.** ______________

2021 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

*****Calculators may not be used on the first four questions*****

1. Let $\begin{vmatrix} 7 & 0 & 4 \\ x & 5 & -2 \\ -1 & 3 & 0 \end{vmatrix} = 158$ and $y = \dfrac{\log_4(27)}{\log_2(9)}$. Find the product xy.

2. Let 36, k, and 20 in that order be the terms in an arithmetic sequence. Let 12, w, and 48 in that order be terms in a geometric sequence with $w > 0$. Then the terms k, w, and p, in that order form another arithmetic sequence. Determine the value of p.

3. The point P is located on the x-axis and the point Q is located on the y-axis. Each point is 5 units from the point (3,4). Determine the largest possible length of $\overline{PQ}$.

4. Let k be the perimeter of right triangle ΔABC having right angle at C, $\angle A \cong \angle B$ and $AC = 3$. Let w be the area of parallelogram $DFGH$ has $DF = 4$, $DH = 6$, and $\angle F = 120°$. Find the sum $k + w$. Write your answer as a radical expression in the form $p + q\sqrt{r} + s\sqrt{t}$ where all constants represent positive integers and r and t are as small as possible.

*****Calculators may be used on the remaining questions*****

5. Ten lines lie in a plane in such a way that no two lines are parallel and no point of intersection contains more than two lines. Determine the largest number of regions into which these 10 lines divide the plane.

6. Let $f(x) = \begin{cases} 5 - x & \text{if } x < 2 \\ |x - 9| & \text{if } x \geq 2 \end{cases}$

   Determine the sum of all values for x such that $f(x) = 6$.

7. Let $20^x = 16$ and $16^y = 20$. Determine the value of $\dfrac{20^{(x+2)}}{16^{(2y-1)}}$.

8. An antenna is mounted on the outer edge of the roof of a building. From a point on ground level 30 feet away from that side of the building, the angles of elevation to the bottom and top of the antenna are 37° and 51° respectively. Determine the distance, in feet, from the bottom to the top of the antenna. Express your answer as a decimal rounded to the nearest tenth of a foot.

9. (Tiebreaker #1) All angles are measured in radians. Determine the value of the sum

   $\sin(0) + \sin\left(\dfrac{\pi}{4}\right) + \sin\left(\dfrac{2\pi}{4}\right) + \sin\left(\dfrac{3\pi}{4}\right) + \cdots + \sin\left(\dfrac{10000\pi}{4}\right)$

10. (Tiebreaker #2) Ann and Mark each buy a certain item for the same price. Ann then sells hers for a 10% profit, while Mark sells his for a 10% loss. Ann's sale price is $10 more than Mark's sale price. Determine the number of dollars in the original price they each paid for the item.

Names: ______________________________ School: ______________________

2021 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

		SCORE
1.	____________________________	_____
2.	____________________________	_____
3.	____________________________	_____
4.	____________________________	_____
5.	____________________________	_____
6.	____________________________	_____
7.	____________________________	_____
8.	____________________________	_____
T1.	____________________________	
T2.	____________________________	

Calculators are not allowed to be used on the first four questions!

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE

2021 John O'Bryan Mathematics Competition
5-Person Team Test

1. (a) Find prime numbers $p < q < r < s$ satisfying $p^q \cdot r \cdot s = 12008$ and $q \cdot r^p = 1083$

> **Solution:** Since 12008 is even, 2 is a factor; factoring 2s gives $12008 = 2^3 \cdot 1501$, which means $p = 2$ and $q = 3$. Since the sum of digits of 1083 is divisible by 3, 1083 is divisible by 3; factoring 3 gives $1083 = 3 \cdot 361$. Since $q = 3$ and $p = 2$, $361 = r^2$ implies $r = 19$. Hence $s = \frac{1501}{19} = 79$. Therefore $p = 2$, $q = 3$, $r = 19$, and $s = 79$ giving $12008 = 2^3 \cdot 19 \cdot 79$ and $1083 = 3 \cdot 19^2$.

(b) Consider 120308, 1203308, 12033308, and 120333308. Which are divisible by r from the previous part?

> **Solution:** From Part (a), we know that $r|12008$. If $120308 - 12008$ is a multiple of r, then r also divides 120308; notice that $120308 - 12008 = 1083$, which is divisible by r by Part (a). Hence $r|120308$.
>
> Notice $1203308 - 120308 = 1083000 = 1083 \cdot 10^3$, which is divisible by r since 1083 is divisible by r. Hence $r|1203308$ Similarly, $r|12033308$ because $12033308 - 1203308 = 1083 \cdot 10^4$, and $r|120333308$ because $120333308 - 12033308 = 1083 \cdot 10^5$. Therefore, all of the considered numbers are divisible by r from the previous part.

(c) Is $120\underbrace{33\ldots3}_{2021}08$ divisible by r from the first part of this question?

> **Solution:** Noticing the pattern formed in the solution to previous part of this question, $120\underbrace{33\ldots3}_{n \text{ threes}}08 - 120\underbrace{33\ldots3}_{n-1 \text{ threes}}08 = 1083 \cdot 10^{n+1}$. By the same logic, all numbers of the form $120\underbrace{33\ldots3}_{n threes}08$ are divisible by r.

2. A *ribbon of length* k is a rectangle with dimensions $1 \times k$ for some natural number k.

(a) Consider covering areas with ribbons of distinct lengths; that is, every ribbon used to cover the area has a different length. What is the maximum area that can be covered if the longest ribbon has length $k = 5$? $k = 10$? $k = 15$? $k = 20$? general k?

> **Solution:** The maximum area is attained when a ribbon of every possible length is used. In general, $1 + 2 + \cdots + k = k(k+1)/2$. This gives $5 \cdot 6/2 = 15$, $10 \cdot 11/2 = 55$, $15 \cdot 16/2 = 120$ and $20 \cdot 21/2 = 210$

(b) If the longest ribbon has length 15, what is the maximum value of m such that an $m \times 15$ rectangle can be covered using ribbons of distinct lengths?

> **Solution:** From the previous question, the maximum area that can be covered is 120. With one dimension equal to 15, $m \le 120/15 = 8$. Notice that 8 rows can be formed with ribbons of distinct lengths as follows: $15 + 0 = 14 + 1 = 13 + 2 = 12 + 3 = \cdots = 8 + 7$. Thus the maximum such m is 8.

(c) Assume $m \ge n$. How large must m be so that an $m \times n$ rectangle can be covered using ribbons of distinct lengths?

> **Solution:** Because m is the length of the longest dimension, the longest ribbon possible has length m. From the first part of this question, the maximum area that can be covered is $m(m+1)/2$, which means the area of the rectangle $m \cdot n \le m(m+1)/2$. Hence $n \le (m+1)/2$ or $m \ge 2n - 1$. It remains to show that an $(2n-1) \times n$ rectangle can be covered.
>
> If m is odd, then (as with the previous part) $m = (m-1) + 1 = (m-2) + 2 = \cdots (m+1)/2 + (m-1)/2$ gives $1 + (m-1)/2 \ge 1 + ([2n-1] - 1)/2 = n$ rows of length m. If m is even, then (similar to the previous part) $m = (m-1) + 1 = (m-2) + 2 = \cdots (m/2-1) + (m/2+1)$ gives $1 + (m/2-1) \ge 1 + ([2n-1]/2 - 1) = n + 0.5 \ge n$ rows of length m.

3. (a) For real numbers x and y, show that $|x+y| \leq |x| + |y|$. (Hint: if $a, b \geq 0$ with $a^2 \leq b^2$, then $a \leq b$.)

> **Solution:** Recall $x \leq |x|$ for all real numbers x and $x^2 = |x|^2$. Notice $xy \leq |x||y|$ so $x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$, which is equivalent to $(x+y)^2 \leq (|x| + |y|)^2$ where the left hand side equals $|x+y|^2$.

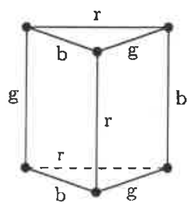
(b) For real numbers x, y, and z, show that $|x+y+z| \leq |x+y-z| + |x-y+z| + |-x+y+z|$.

> **Solution:** Using the previous part, we have $|x+y-z| + |x-y+z| \geq |(x+y-z) + (x-y+z)| = 2|x|$. Similarly $|x-y+z| + |-x+y+z| \geq 2|z|$ and $|-x+y+z| + |x+y-z| \geq 2|y|$. Adding up the three inequalities and dividing by 2 gives the desired result.

4. Consider coloring the edges of a prism using 3 colors, say red (r), blue (b), and green (g). A face of the prism "sees" a color if (at least) one of the edges along the border of the face is colored with that color. A vertex of the prism "sees" as color if (at least) one of the three edges touching the vertex is colored with that color.

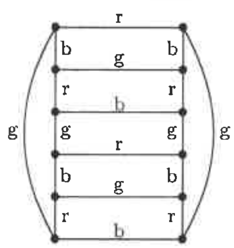
   (a) Can a triangular prism be colored so that every face and every vertex see all 3 colors?

   Solution: Yes.

   (b) Can a hexagonal prism be colored so that every face and vertex see all 3 colors?

   Solution: Yes.

   (c) Can a rectangular prism be colored so that every face and vertex see all 3 colors?

   Solution: No. Suppose such a coloring exists. Under such a coloring, every vertex has exactly one incident edge of each color. Consider the "top" face of the prism. Since all three colors appear on edges and there are four total edges, we can find three consecutive edges of distinct colors, say r, b, g from left to right. Since the two vertices in the middle have two colors, the colors on the remaining "downward" edge at those vertices are determined to be g and r (left and right respectively). This forces the bottom edge under the b-colored top edge to be colored b, which in turn determines the colors for remaining edges at the two bottom-middle vertices (r and g for left and right respectively). This forces the remaining "vertical" edges (incident to the initial r and g colored edges on top) to both be colored b. The remaining edge incident to the left-most top vertex must be colored g and the remaining edge incident to the right-most top vertex must be colored r. However, those are the same edge and it is impossible to color a single edge with two colors. Therefore, it is impossible for such a coloring to exist.

5. The integers from 1 to n are written in a row. The same numbers are written under them in a different order (see below). Note that there are no repeated numbers in a row.

1	2	3	4	5
4	2	1	5	3

Notice that the sum of numbers in the middle three columns is a perfect square. Is it possible that the sums of columns are all perfect squares when

(a) $n = 9$?

Solution: Yes. The sum of the number 9 and the number beneath it lies between 10 and 18. Since 16 is the only perfect square in this interval, the number under 9 must be 7. Similarly, the number 7 must be written above the number 9 on the second row. Similarly, the numbers under 4, 5, and 6 must be 5, 4, and 3 (respectively). The table can be completed as follows:

1	2	3	4	5	6	7	8	9
8	2	6	5	4	3	9	1	7

(b) $n = 11$?

Solution: No. Using the same reasoning in the previous part, the only number that can be written under 11 is 5. However, the number under 4 must also be 5, which is impossible.

(c) $n = 1992$?

Solution: The first square past 1992 is $2025 = 45^2$, which equals $1992 + 33$. So if under each number k from 33 through 1992 we write the number $2025 - k$, the "sum is a square" condition is satisfied for all columns after the first 32. Note that we have not yet written the numbers 1 through 32 in the bottom row.

Repeating the same strategy we write under each number k from 4 through 32 its counterpart $36 - k$. Under the remaining numbers in the top row of 1, 2, and 3, we write the remaining numbers of 3, 2, and 1 (respectively) in the bottom row.

6. The side BC of an equilateral triangle ABC is divided into three equal parts by the points K and L with K closer to B. The point M divides the side AC in the ratio $AM : MC = 1 : 2$.

   (a) Prove ΔMKC is equilateral.

   > **Solution:** Since $MK = KC$ and $\angle MCK = 60°$, $\Delta MKC \sim \Delta ABC$. Since ΔABC is equilateral, so is ΔMKC.

   (b) Prove $\overline{AB} \parallel \overline{MK}$.

   > **Solution:** This follows from $\angle MKC = \angle ABC = 60°$ and the fact that $\overline{CB}$ and $\overline{CK}$ are colinear.

   (c) Prove $\angle AKM = \angle CAL$.

   > **Solution:** Notice that side-angle-side with $\angle ABK$ and $\angle ACL$ implies $\Delta BAK \cong \Delta CAL$, which means $\angle BAK = \angle CAL$. Since $\overline{AB} \parallel \overline{MK}$, $\angle AKM = \angle BAK$ as alternate interior angles. Combining these two gives the desired result.

   (d) Prove $\angle AKM + \angle ALM = 30°$.

   > **Solution:** Using the previous part, notice that $\angle AKM + \angle ALM = \angle CAL + \angle ALM$. Since $\angle AML$ and $\angle LMC$ are supplementary angles, $180° = \angle LMC + \angle AML = \angle LMC + (180 - \angle CAL - \angle ALM)$, where the second follows from the angles being in ΔAML. Thus $\angle AKM + \angle ALM = \angle LMC$. Since $\overline{ML}$ is a median in the equilateral triangle MKC, $\overline{ML}$ is an angle bisector of $\angle KMC = 60°$. Therefore $\angle AKM + \angle ALM = \angle LMC = \frac{1}{2}60° = 30°$, as desired.

Name: ______ANSWERS______ Team Code: _______

2021 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 5
2. 20
3. 4.5 — Must be this decimal.
4. 60
5. (10,17) — Must be this ordered pair.
6. 2/15 — Must be this reduced fraction.
7. $\frac{2636}{495}$ — Must be this improper fraction.
8. 276
9. 5
10. 10
11. 1/8 — Must be this reduced fraction.
12. 9
13. 200
14. 32
15. 1 and -1/3 — Must have both answers, in either order
16. 8.677 — Must be this exact decimal.
17. $2\sqrt{13}$ — Must be this radical expression
18. 27/34 — Must be this reduced fraction.
19. 3
20. 503

Awards Lists and Solutions to the Team Competition may be found at https://inside.nku.edu/artsci/departments/math/competitions/obryan.html

Name: ______ANSWERS______ Team Code: ________

2021 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 2

2. $2\sqrt{13}$ — Must be this radical expression.

3. 96

4. 2

5. -6

6. $5/6$ — Must be this reduced fraction.

7. $5/3$ — Must be this improper fraction.

8. $-1/2$ — Must be this reduced fraction.

9. 137

10. 2

11. $-24/25$ — Must be this reduced fraction.

12. 325

13. $\frac{3}{4}b\sqrt{a}$ — Must be this exact answer.

14. 503

15. 41

16. $1/8$ — Must be this reduced fraction.

17. 406

18. $1/5$ — Must be this reduced fraction.

19. 88.47 — Must be this exact decimal.

20. $\left(\frac{739}{83}, \frac{110}{13}\right)$ — Must be this ordered pair.

Names: ______________________________ **School:** ______________________

2021 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

Calculators are not allowed to be used on the first four questions!

	Answer		SCORE
1.	6		____
2.	20		____
3.	10		____
4.	$6+3\sqrt{2}+12\sqrt{3}$	Terms may be in any order	____
5.	56		____
6.	17		____
7.	256		____
8.	14.4	Must be exactly this decimal.	____
T1.	0		
T2.	50		

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE