Directions: You may use your (hand-written) notes, and a graphing calculator (or the equivalent).

Each problem should be attempted on a separate sheet of paper, with the subparts carried out in order (you may leave a space if you want to skip over a part). This is to make it easier for me to grade (remember, keep your grader happy!). Extra plots are on the last page, if you want to use them.

Each subpart of the test problems is equally weighted. Do your best, and **good luck!**

Preliminaries

You'll be working with the function $f(x) = x(x-1)e^x$. Below are given the first three derivatives, along with the roots of the derivative, and a graph of the function.

You will need to understand the asymptotic behavior of the function f (that is, its behavior as x goes to plus or minus infinity). You may obviously plot it, to check your understanding.

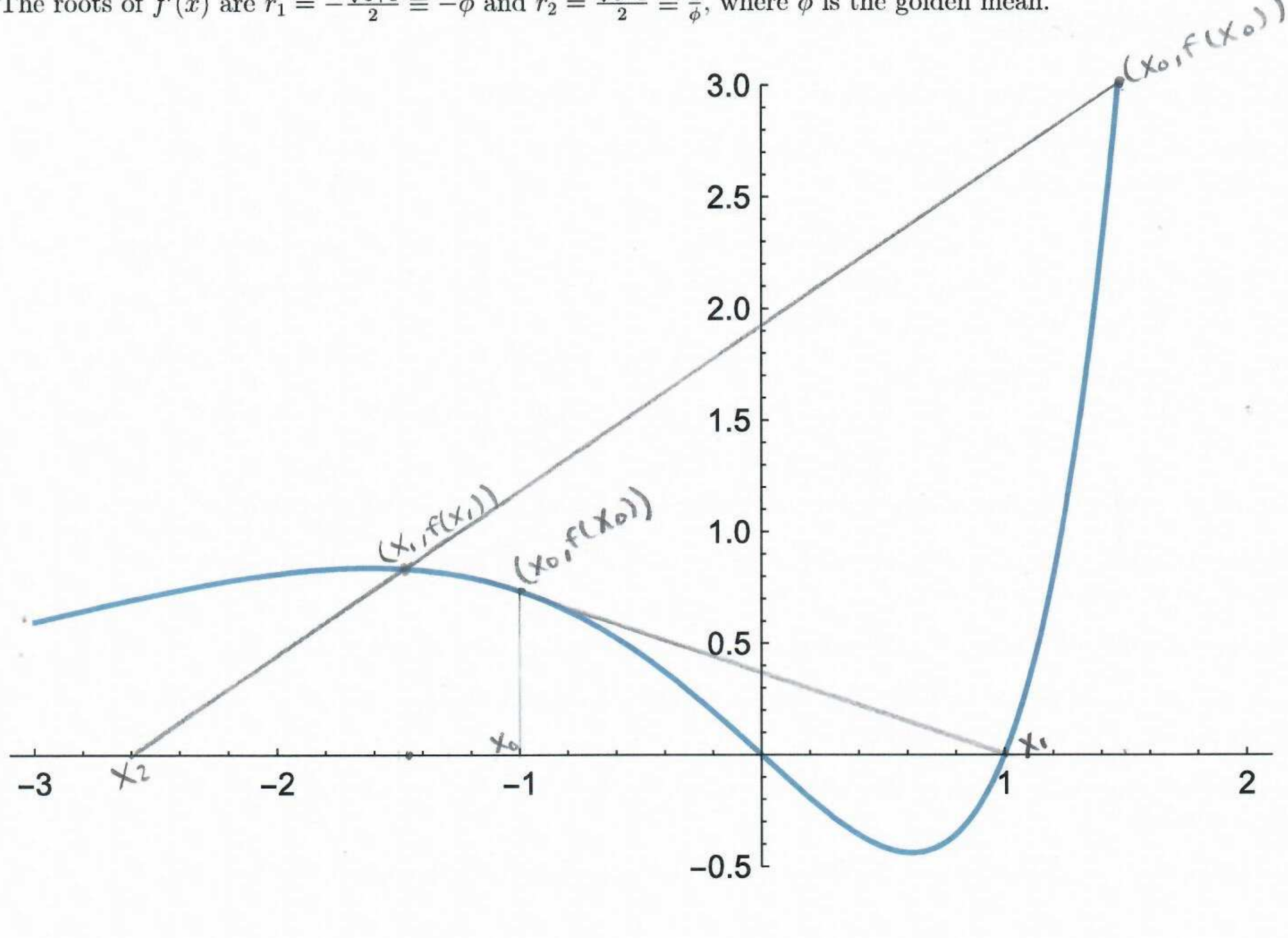
$$f(x) = e^x \left(x(x-1)\right)$$

$$f'(x) = e^x \left(x^2 + x - 1\right)$$

$$f''(x) = e^x \left(x^2 + 3x\right)$$

$$f'''(x) = e^x \left(x^2 + 5x + 3\right)$$

The roots of $f'(x)$ are $r_1 = -\frac{\sqrt{5}+1}{2} \equiv -\phi$ and $r_2 = \frac{\sqrt{5}-1}{2} \equiv \frac{1}{\phi}$, where ϕ is the golden mean.



$(x) = x(x-1)e^x$

ⓐ $g'(x) \neq 0$

so $x_0 = -\frac{\sqrt{5}+1}{2} \equiv -\phi$ and $x_0 = \frac{\sqrt{5}-1}{2} \equiv \frac{1}{\phi}$

will cause Newton's Method to crash

ⓑ $(-\infty, -\frac{\sqrt{5}+1}{2})$ is the largest intervall for x_0 that will cause Newton's Method to fail to converge. The tangent lines for any x_0 on this interval will keep shooting backwards toward $-\infty$ since the slopes at these points are positive

good

ⓒ yes, there is an interval around the left side of the minimum $\left(\frac{\sqrt{5}-1}{2}, f\left(\frac{\sqrt{5}-1}{2}\right)\right)$ that will land in the interval $(-\infty, -\frac{\sqrt{5}+1}{2})$, which means Newton's Method will eventually di converge for this interval as well (or fail to converge)

ⓓ $x_0 = 0$ and $x_1 = 1$ the secant method will fail because $f(0) = f(1)$

$x_0 = 1.5$ $x_1 = 2$ will clearly lead to root $x = 1$, the tangent line will shoot down to the axis at a point very close to $x=1$, and then will keep repeating until it converges to $x = 1$

Yes, clearly! :)

ⓔ perform one step of Newton's Method with $x_0 = -1$

$x_0 = -1$

$x_1 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{2\cancel{e^{-1}}}{-\cancel{e^{-1}}} = -1 + 2 = 1$ (Do you feel lucky?)

ⓕ Not off to a good start, looks like secant method will keep shooting off in negative direction and diverge.

Good pictures

(g) Newton's Method converges much faster than any other method. Newton's method converges quadratically as long as g is twice differentiable and $g'(\hat{x}) \neq 0$

f -

in particular, $g'(r)=0$

(h) $f(x) = x(x-1)e^x$

g is "usually" the FPI function, created from f.

$x(x-1)e^x = 0$

$x(x-1)e^x + x = x$

✓ $\boxed{g(x) = x(x-1)e^x + x}$ $\quad e^x x^2 - \cancel{e^x x} + x$

$g'(x) = e^x x^2 + 2e^x - e^x x - e^x + 1$

$= e^x(x^2 + 2 - x - 1) + 1$

$= e^x(x^2 - x + 1) + 1$

$e^x(x^2 - x + 1) + 1 < 1$

We need to bound $g'(x)$ in the vicinity of 1, so we need $|g'(1)| < 1$

abs. values

$e^x(x^2 - x + 1) < 0$

$e^x(x^2 - x + 1$ is always positive so no value of x satisfies

This tells us the derivative is never less than 1, so there is no x_0 that is guaranteed to converge to any root of $f(x)$

Nice work

So no, mine will not converge to root $x=1$ even if we start in the vicinity of $x=1$.

(i) The interval $[-1, 1.1]$ does not guarantee convergence of the bisection method because $f(-1)$ and $f(1.1)$ are both positive, so the assumptions of IVT are not met. But the next step produces interval $[-1, 0.05]$ which does satisfy the conditions of Bisection because $f(-1)$ is positive and $f(0.05)$ is negative. So the IVT guarantees a value c such that $f(c)=0$. ✓

Problem 2:

a. $p_n = \frac{3}{2}$ $p_{n+1} = \frac{5}{3}$ $p_{n+2} = \frac{8}{5}$

$$a_2 = \frac{3}{2} - \frac{\left(\frac{5}{3} - \frac{3}{2}\right)^2}{\frac{8}{5} - 2 \cdot \frac{5}{3} + \frac{3}{2}}$$

$a_2 = 1.6190476 = \frac{34}{21}$

b.

$$\frac{\left|\frac{34}{21} - \frac{1+\sqrt{5}}{2}\right|}{\frac{1+\sqrt{5}}{2}} = 0.00062645798$$

$$\frac{\left|\frac{8}{5} - \frac{1+\sqrt{5}}{2}\right|}{\frac{1+\sqrt{5}}{2}} = 0.11146$$

0.011...

not even close, aitkens way better

c. aitkens

$$\alpha \approx \frac{\ln\left|\frac{p_{n+2} - p}{p_{n+1} - p}\right|}{\ln\left|\frac{p_{n+1} - p}{p_n - p}\right|}$$

$p = \frac{1+\sqrt{5}}{2}$

$p_n = \frac{5}{3}$

$p_{n+1} = \frac{13}{8}$

$p_{n+2} = \frac{34}{21}$

$\alpha = 0.90189$

r (regular)

$p = \frac{1+\sqrt{5}}{2}$

$p_n = \frac{3}{2}$

$p_{n+1} = \frac{5}{3}$

$p_{n+2} = \frac{8}{5}$

$\alpha = 1.11882627$

d.

$$S_0 = a_n - \frac{(a_{n+1} - a_n)^2}{a_{n+2} - 2a_{n+1} + a_n}$$

$a_n = \frac{5}{3}$

$a_{n+1} = \frac{13}{8}$

$a_{n+2} = \frac{34}{21}$

$S_0 = 1.61805556$ ✓

e.

error relative is equal to

0.0000133290l

It is far better than just 1 aitkens, but is not getting better that much faster, in fact, compared to the first 2 it feels like it is getting better linearly, gaining 2 digits every time, which makes me feel like it is not worth, especially considering how many calculations are required to get here.

Excellent.

3.) $e^1(1(1-1)) + \frac{e^1(1^2+1-1)}{1!}(x-1) + \frac{e^1(1^2+3(1))}{2!}(x-1)^2$

$= 0 + \frac{2.718281828}{1!}(x-1) + \frac{10.87312731}{2!}(x-1)^2$

or $\frac{e^1}{1!}(x-1) + \frac{e^1(4)}{2!}(x-1)^2$ $= e(x-1) + 2e(x-1)^2$

Simplify when its easy

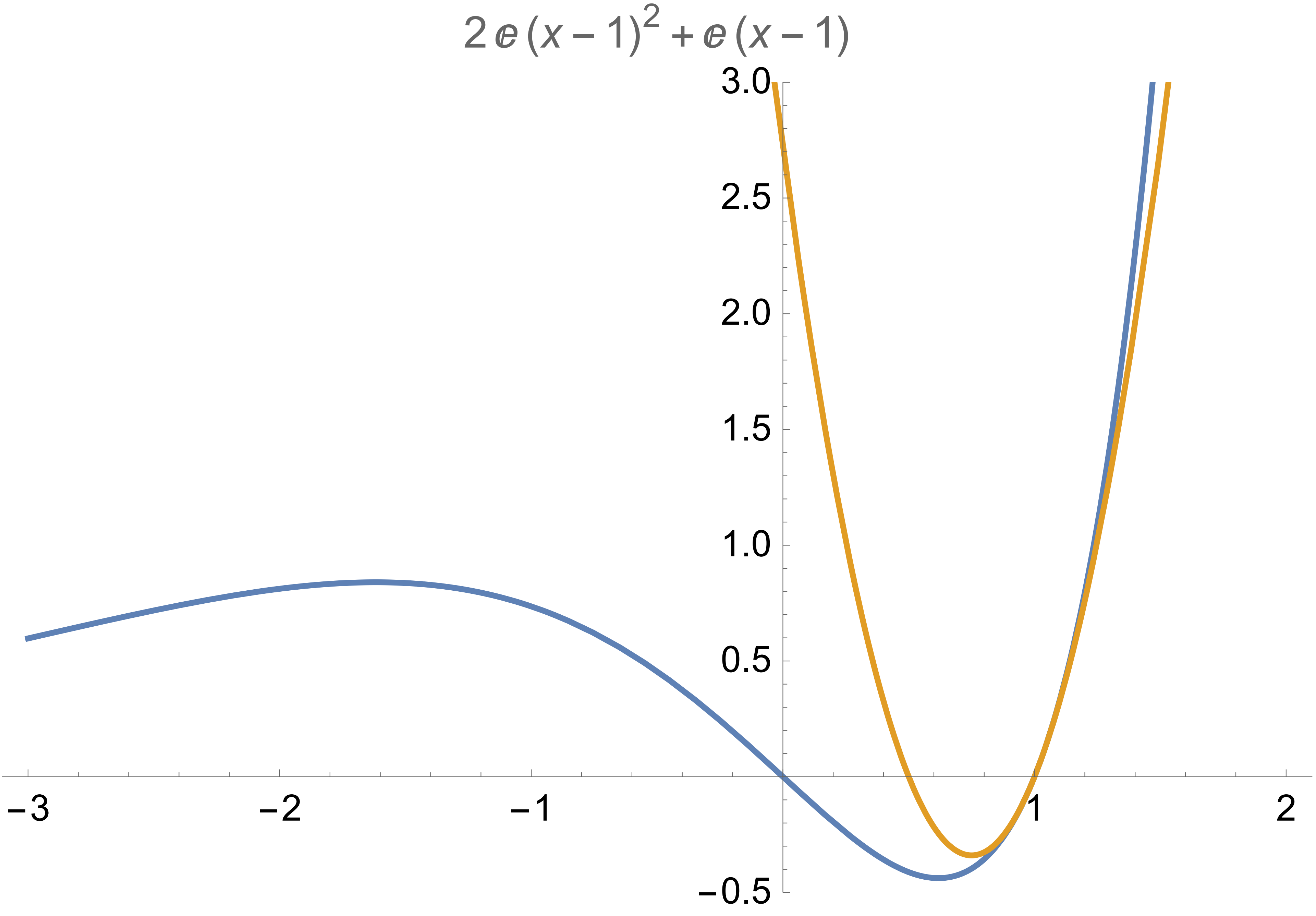
b.) $R_2(x) = \frac{e^{\xi(x)}(\xi(x)^2 + 5(\xi(x)) + 3)}{3!}(x-1)^3$ ✓

Justify

d.) $\frac{e^1(1^2+5(1)+3)}{3!}(0-1)^3 = 4.077422743$

e.) The absolute error on the interval is 3 which 4.077422743 is larger. ✓

e



$2e(x-1)^2 + e(x-1)$
3.0
2.5
2.0
1.5
1.0
0.5
−0.5
−3
−2
−1
1
2