A quadratic doesn't have to look like $ax^2 + bx + c$

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Yeah, usually when someone says something about a quadratic the next breath contains $ax^2 + bx + c$. But let's take a look at how best to write the quadratic that fits three points.

There are several contenders, but I want to share just one with you today. Consider the three points that we want to fit with quadratic q: $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ with distinct abscissa (x values) and assume that they are non-linear (so can't be fit with a simpler polynomial – that q is a true quadratic).

Here's how one might do it:

$$q(x) = y_3 + (x - x_3)[\beta + (x - x_3)\alpha]$$

Notice that we've found one of our constants already, and have only two left to determine.

Check: $q(x_3) = y_3$ – so it fits one of the three points **by design**.

Now we use the other two points to create two equations in the two unknowns α and β:

$$\begin{aligned} y_3 + (x_2 - x_3)[\beta + (x_2 - x_3)\alpha] &= y_2 \\ y_3 + (x_1 - x_3)[\beta + (x_1 - x_3)\alpha] &= y_1 \end{aligned}$$

which we re-write as

$$\begin{aligned} \beta + (x_2 - x_3)\alpha &= \tfrac{y_2 - y_3}{(x_2 - x_3)} \\ \beta + (x_1 - x_3)\alpha &= \tfrac{y_1 - y_3}{(x_2 - x_3)} \end{aligned}$$

Defining $d_{ij} \equiv x_i - x_j$ and $f_{ij} \equiv \dfrac{y_i - y_j}{x_i - x_j}$, we re-write these as

$$\begin{aligned} \beta + d_{23}\alpha &= f_{23} \\ \beta + d_{13}\alpha &= f_{13} \end{aligned}$$

Then, subtracting the bottom equation from the top, we get

$$\alpha = \frac{f_{23} - f_{13}}{d_{21}}$$

Define

$$f_{ijk} \equiv \frac{f_{ij} - f_{jk}}{d_{ik}}$$

so that

$$\alpha = \frac{f_{23} - f_{13}}{d_{21}} = \frac{f_{23} - f_{31}}{d_{21}} \equiv f_{231}$$

(notice that $f_{ij} = f_{ji}$, by definition). Then

$$\beta = f_{23} - d_{23}\alpha$$

Finally

$$q(x) = y_3 + (x - x_3)\beta + (x - x_3)^2\alpha$$

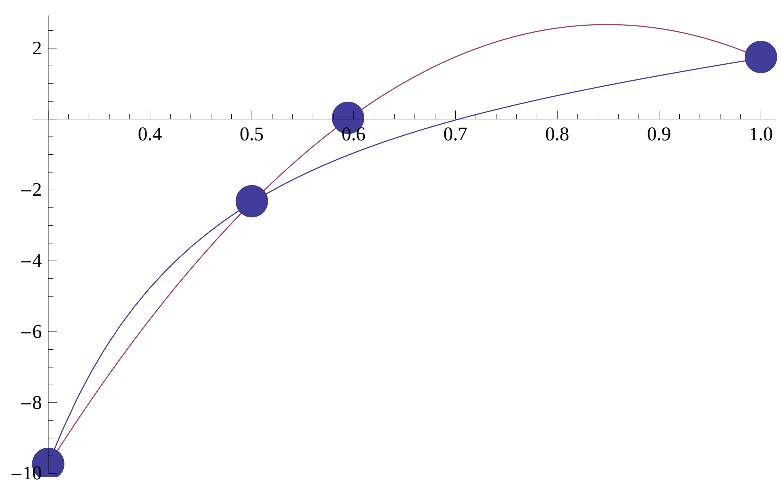
and we can use the quadratic formula to solve for values of $(x - x_3)$ that make $q(x) = 0$:

$$x - x_3 = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha y_3}}{2\alpha}$$

or

$$x = x_3 + \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha y_3}}{2\alpha} \tag{1}$$

So we can use this technique when doing such curve-fitting, such as when we use a quadratic passing through three points to approximate a root of a given function in Muller's method: Can you tell which is the quadratic and which is the given function?



Note that, computationally at least, the method of (1) may be suboptimal in Muller's method. A better form of the quadratic equation to find the root closest to x_3 (assumed to be the most recently obtained) is

$$x = x3 - \frac{2y_3}{\beta + \sqrt{\beta^2 - 4\alpha y_3}},$$

if $\beta > 0$, and

$$x = x3 - \frac{2y_3}{\beta - \sqrt{\beta^2 - 4\alpha y_3}},$$

otherwise.