Homework 2.3: Tea time

Problems, pp. 67--, #3, 10, 13, 15, 16, 24.

Exercise #3, p. 67

3. Write an Octave function that implements Steffensen's method.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Written by Dr. Len Brin         15 January 2014 %
% Modified by Andy Long           6 February 2024 %
% Purpose: Implementation of Steffensen's method  %
% INPUT: function f; initial value x0; tolerance  %
%        TOL; maximum iterations N0               %
% OUTPUT: approximation x and number of           %
%        iterations i; or message of failure      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [x,i] = steffensen(f,x0,TOL,N0)
  i=1;
  A=x0;
  B=f(A);
  while (i<=N0)
    x0=A;
    x1=B;
    x2=f(x1);
    if (abs(x2-x1)<TOL)
      x=x2;
%      disp(" ");
      return
    end%if
    x=x0-(x1-x0)^2/(x2-2*x1+x0);
%    disp(x);
    if (abs(x-x2)<TOL)
%      disp(" ");
      return
    end%if
    A=x;
    B=f(A);
    i=i+1;
  end%while
  x="Method failed---maximum number of iterations reached";
end%function
```

%%

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function [x,i] = steffensen(f,x0,TOL,N0)
 i=1;
 A=x0;
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 while (i<=N0)
  x0=A;
  x1=B;
  x2=f(x1);
  if (abs(x2-x1)<TOL)
   x=x2;
   return
  end%if
  x=x0-(x1-x0)^2/(x2-2*x1+x0);
  if (abs(x-x2)<TOL)
   return
  end%if
  A=x;
  B=f(A);
  i=i+1;
 end%while
 x="Method failed---maximum number of iterations reached";
end%function
```

This Mathematica version of Steffensen's not only computes using Steffensen's acceleration, but it also compares the result to

* the standard FPI using the given function, and
* the Aitken's acceleration sequence that goes along with it.

```
In[132]:= steffensen[g_, q0_, tol_, n_] := Module[{i = 0, a, x1, x2, x0, y1, y2, y0, denom},
   (* the x sequence will be Steffensen's *)
   x0 = q0; (* Set x0 to the initial guess, and compute two iterates:  *)
   x1 = N[g[x0]];
   x2 = g[x1];
   y0 = x0;  (*The y's will be regular fpi: *)
   y1 = x1;
   y2 = x2;
   (* We've constructed the first two iterates from q0 *)
   denom = (x2 - 2 * x1 + x0);
   Print[
    "We produce comparable results after each two iterations of the FPI:"
   ];
   Print[{"FPI", "Aitken's", "Steffensen's"}];
   Print[{y0, "NA", "NA"}];
   Print[{y1, "NA", "NA"}];
   While[Abs[denom] > tol && i < n,
    (* reset initial starting values for the sequences: *)
    x0 = x0 - (x1 - x0) ^ 2 / denom;
    a = y2 - (y1 - y2) ^ 2 / (y2 - 2 * y1 + y0);
    y0 = y2;
    Print[{y0, a, x0}];
    y1 = g[y0];
    y2 = g[y1];
    x1 = g[x0];
    x2 = g[x1];
    denom = (x2 - 2 * x1 + x0);
    x0 = x2;
    i = i + 1
   ];
   x0
  ]
steffensen[Cos, 0.999, 10^-8, 30]
NSolve[Cos[x] == x, x, Reals]
```

```
We produce comparable results after each two iterations of the FPI:
{FPI, Aitken's, Steffensen's}
{0.999, NA, NA}
{0.541143506561572, NA, NA}
{0.857120202838969, 0.728098504212883, 0.728098504212883}
{0.793281060653079, 0.736922517293844, 0.739067254325221}
{0.763868018369208, 0.738639413517506, 0.739085133167616}
```

Out[133]= 0.739085133193587

Out[134]= {{x → 0.739085133215161}}

Exercise #10, p. 67

10. Fixed point iteration on the function $g(x) = \sqrt[3]{x^2 + x}$ will converge to approximately 1.618033988749895 for any x_0 in $[0.5, 3.5]$. [A]

    (a) Find a bound on the number of iterations it will take to achieve 10^{-4} accuracy with $x_0 = 2.5$.

    (b) How many iterations does it actually take to achieve 10^{-4} accuracy with $x_0 = 2.5$?

fixedPoint code:

In[190]:=
```
(* These are options that one can give at game time:
  -- epsilon: the value we consider "zero" (0 by default)
      --
   tolerance: the width of the interval we consider "close enough" to the root
              -- verbose: print the details of the convergence
      -- debug: print out a little more for the coder....
*)
Options[fixedPoint] =
  {maxits → 100, epsilon → 0, tolerance → 10 ^ (-6), verbose → True, debug → False};

(* Here's the function itself: *)
fixedPoint[f_, a0_, OptionsPattern[]] :=

 Module[
  {a = a0, c, fa = f[a0], fc, slope, width, output, i = 1, its = OptionValue[maxits],
   eps = OptionValue[epsilon], tol = OptionValue[tolerance],
   verbose = OptionValue[verbose], debug = OptionValue[debug]},
  width = 10 000;
  (* Update and continue: let's start looping: *)
  While[i ≤ its && width > tol,
   (* c is the function value at a: *)
   c = fa;
   fc = f[c];
   width = Abs[c - a];
   slope = (fc - fa) / (c - a);
```

```
    (* Otherwise, store output and do it again: *)
    If[i == 1,
     (* Let's collect the output,
     since we've gone to the trouble of computing it: *)
     output = {{a, c, fc, slope}},
     output = Append[output, {a, c, fc, slope}]
    ];

    (* Did we stomp on a root by good fortune? *)
    If[width ≤ tol,
     If[verbose,
      Print["Root at c."];
      printoutFixedPoint[output, a, c, fc, width, slope];
      Return[Transpose[output]],
      Return[fc] (* if we're converging, this is better than c *)
     ]
    ];

    (* update: *)
    a = c;
    fa = fc;
    i = i + 1;
   ];

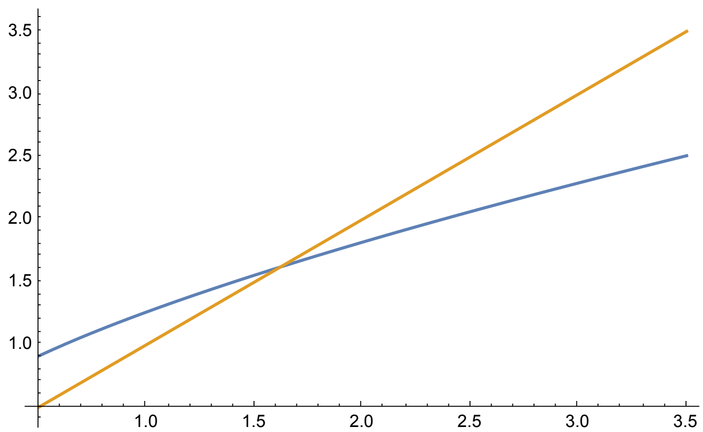
   (* We've popped out, so print some nice formatted output: *)
   If[verbose,
    printoutFixedPoint[output, a, c, fc, width, slope];
    (* Return the output, so that it can be plotted, etc. *)
    Return[Transpose[output]],
    Return[fc]
   ];
  ]
```

This is the printout command for verbose results:

```
In[137]:= printoutFixedPoint[output_, a_, c_, fc_, width_, slope_] :=
  Module[{a0 = output[[1]][[1]]},
   Print[a0];
   Print[NumberForm[TableForm[output,
      TableHeadings → {None, {"ak", "ck", "f[ck]", "slope"}}], 16]];
   Print["  c   = ", NumberForm[c, 16] ];
   Print[" Δc   = ±", width];
   Print["f[c] = ", NumberForm[fc, 16] ];
   Print["f'[c]  = ", NumberForm[slope, 16] ];
  ]

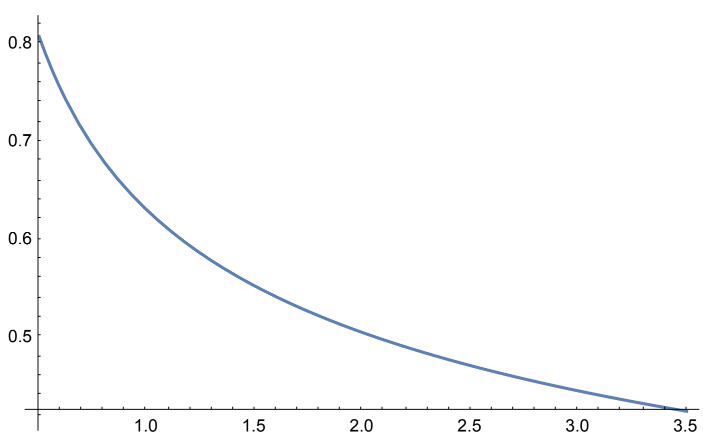
In[192]:= Clear[tolerance];
f[x_] := CubeRoot[x^2 + x]
Plot[{f[x], x}, {x, 0.5, 3.5}]
f'[x]
Plot[f'[x], {x, 0.5, 3.5}]
results = fixedPoint[f, 2.5, maxits → 100, verbose → True, tolerance → 10^-4];
NSolve[f[x] == x, x, Reals]
```

Out[194]=



Out[195]= $\dfrac{1 + 2\,x}{3\,\left(\sqrt[3]{x + x^2}\right)^2}$

Out[196]=



```
Root at c.

2.5
```

a_k	c_k	f[c_k]	slope
2.5	2.060642649904278	1.847587874624534	0.48492366232937
2.060642649904278	1.847587874624534	1.739244930355744	0.508521548632431
1.847587874624534	1.739244930355744	1.682663228881129	0.522246297223016
1.739244930355744	1.682663228881129	1.652676253198925	0.5299765631059592
1.682663228881129	1.652676253198925	1.636655768363659	0.5342481017441409
1.652676253198925	1.636655768363659	1.628059450909683	0.5365828526645954
1.636655768363659	1.628059450909683	1.623435910608436	0.5378512748047227
1.628059450909683	1.623435910608436	1.620945958023321	0.5385380948108121
1.623435910608436	1.620945958023321	1.619604099371818	0.5389093188055789
1.620945958023321	1.619604099371818	1.618880690266264	0.5391097674443019
1.619604099371818	1.618880690266264	1.618490615094499	0.5392179456542331
1.618880690266264	1.618490615094499	1.618280256795016	0.5392763105930964
1.618490615094499	1.618280256795016	1.618166808924338	0.5393077951145521
1.618280256795016	1.618166808924338	1.618105623676694	0.5393247777828193
1.618166808924338	1.618105623676694	1.618072624396149	0.5393339377770753

```
   c  = 1.618105623676694
  Δc  = ±0.0000611852476435359
f[c]  = 1.618072624396149
f'[c] = 0.5393339377770753
```

Out[198]= {{x → 0}, {x → -0.618033988749895}, {x → 1.6180339887499}}

15 iterations; the slope at the fixed point is approximately m=0.5393339377770753, and decreasing to the right; and the initial interval was Abs[2.5-2.060642649904278]=0.439357. So we know that to get to within 10^-4, we need to solve for k, where

m^k*initInterval<tolerance

Mathematica says 14 interations.

```
initInterval = Abs[2.5 - 2.060642649904278];
m = 0.5393339377770753;
tol = 10^-4;
Ceiling[k /. Solve[m^k * initInterval == tol, k]]
```

Out[147]= {14}

Exercise #13, p. 67:

13. Calculate two iterations of Steffensen's method for $g(x) = \sqrt[3]{x^2 + x}$ with $x_0 = 2.5$. [A]

Steffensen's only required 4 iterations:

In[148]:=
```
f[x_] := CubeRoot[x^2 + x]
steffensen[f, 2.5, 10^-4, 30]
```

```
We produce comparable results after each two iterations of the FPI:
{FPI, Aitken's, Steffensen's}
{2.5, NA, NA}
{2.06064264990428, NA, NA}
{1.84758787462453, 1.64700536995611, 1.64700536995611}
{1.68266322888113, 1.62081214554271, 1.61808449770964}
```

Out[149]= 1.61804868132471

Exercise #15, p. 67:

15. Compute a_0, a_1, and a_2 of Aitken's delta-squared method for the sequence in problem 2 on page 29. Since the sequence has an undefined term at $n = 1$, start the sequence $\langle \frac{n+1}{n-1} \rangle$ with $n = 2$. In other words, consider the sequence in problem 2 on page 29 to be $3, 2, \frac{5}{3}, \frac{3}{2}, \frac{7}{5} \ldots$ so $p_0 = 3$, $p_1 = 2$, $p_2 = \frac{5}{3}$, and so on.

In[150]:=
```
aitken[seq_] := Module[{n = Length[seq]},
  TableForm[Table[{seq[[i]], If[i < 3, "NA",
      seq[[i]] - (seq[[i - 1]] - seq[[i]]) ^2 / (seq[[i]] - 2 * seq[[i - 1]] + seq[[i - 2]])]}, {i, 1, n}]
  ]
 ]
seq = Table[(n + 1) / (n - 1), {n, 2, 10}];
aitken[seq]
```

Out[152]//TableForm=

3	NA
2	NA
$\frac{5}{3}$	$\frac{3}{2}$
$\frac{3}{2}$	$\frac{4}{3}$
$\frac{7}{5}$	$\frac{5}{4}$
$\frac{4}{3}$	$\frac{6}{5}$
$\frac{9}{7}$	$\frac{7}{6}$
$\frac{5}{4}$	$\frac{8}{7}$
$\frac{11}{9}$	$\frac{9}{8}$

Exercise #16, p. 67:

16. The following sequences are linearly convergent. Generate the first five terms of the sequence $\langle a_n \rangle$ using Aitken's delta-squared calculation.

    (a) $p_0 = 0.5$, $p_n = (2 - e^{p_{n-1}} + p_{n-1}^2)/3$ for $n \geq 1$ [S]

    (b) $p_0 = 0.75$, $p_n = \sqrt{e^{p_{n-1}}/3}$ for $n \geq 1$

In[153]:=
```
p = {0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0};
seq = Table[p[[n]] = If[n < 2, 0.5, (2 - E^p[[n - 1]] + p[[n - 1]]^2) / 3], {n, 1, 10}]
aitken[seq]
```

Out[154]=
```
{0.5, 0.200426243099957, 0.272749065098375, 0.25360715658413,
 0.258550376264936, 0.257265636335094, 0.25759898516219,
 0.257512454514832, 0.257534913615251, 0.257529084167956}
```

Out[155]//TableForm=

0.5	NA
0.200426243099957	NA
0.272749065098375	0.258684427565791
0.25360715658413	0.25761321071575
0.258550376264936	0.257535832326668
0.257265636335094	0.257530660001032
0.25759898516219	0.257530310659602
0.257512454514832	0.257530287139163
0.257534913615251	0.257530285554338
0.257529084167956	0.257530285447573

In[156]:=
```
p = {0.75, 0, 0, 0, 0, 0, 0, 0, 0, 0};
seq = Table[p[[n]] = If[n < 2, 0.75, Sqrt[E^p[[n - 1]] / 3]], {n, 1, 10}];
aitken[seq]
```

Out[158]//TableForm=

0.75	NA
0.840039684898413	NA
0.878722349663284	0.907858552453487
0.895883434851226	0.909567506867182
0.903603675382265	0.909916893745995
0.907098434998335	0.9099888384861
0.908684866132757	0.910003697648647
0.909405935058004	0.910006770626559
0.909733866349821	0.910007406512369
0.909883043680466	0.910007538129889

Exercise #24, p. 67

24. Find the fixed point of $f(x) = x - 0.002(e^x \cos(x) - 100)$ in $[5, 6]$ using Steffensen's method. [A]

In[211]:=
```
f[x_] := x - 0.002 * (Exp[x] Cos[x] - 100)
result = steffensen[f, 5, 10^-8, 30];
N[result, 17]
```

```
We produce comparable results after each two iterations of the FPI:
{FPI, Aitken's, Steffensen's}
{5, NA, NA}
{5.11580159787492, NA, NA}
{5.18497390946935, 5.28758771879368, 5.28758771879368}
{5.2414055375254, 5.26097834989244, 5.25941275393994}
{5.25516157698512, 5.25928072510421, 5.25918573034734}
```

Out[213]= 5.25918571860258