Homework 1.3

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Section 1.3, pp. 29-30, #2, 4, 6a-c, 9, 12, 19

2. Show that the sequence $\left\langle \frac{n+1}{n-1} \right\rangle$ converges to 1 linearly.

First we can just motivate that it appears to be linearly convergent (α=1).

In[∘]:=
```
a[n_] := (n + 1) / (n - 2)
alpha[a_, p_, n_] :=
 Log[Abs[(a[n + 2] - p) / (a[n + 1] - p)]] / Log[Abs[(a[n + 1] - p) / (a[n + 0] - p)]]
TableForm[Table[{n, N[alpha[a, 0, n]]}, {n, 2, 20}]]
```

Out[∘]//TableForm=

2	0.
3	0.474769847356949
4	0.598410269255311
5	0.671094316687522
6	0.720198264178168
7	0.755985159087087
8	0.783379467375188
9	0.80509372665675
10	0.822763837935186
11	0.837442469891521
12	0.84984100710691
13	0.860459072682846
14	0.869658676126573
15	0.877708978181459
16	0.88481451196434
17	0.891133626879348
18	0.896790915793338
19	0.901885810359314
20	0.906498662848436

4. Give an example of a sequence which converges to 0 with order α=10.

In[∘]:=
```
a[n_] := 2^(1 - 10^n)
TableForm[Table[{n, N[alpha[a, 0, n]]}, {n, 2, 5}]]
```

Out[∘]//TableForm=

2	10.
3	10.
4	10.
5	10.

6. Some linearly convergent sequences and their limits are given. Find the (fastest) rate of convergence of the form $O\left(\frac{1}{n^p}\right)$ or $O\left(\frac{1}{a^n}\right)$ for each. If this is not possible, suggest a reasonable rate of convergence.

   (a) $6, \frac{6}{7}, \frac{6}{49}, \frac{6}{343}, \frac{6}{2401}, \ldots \to 0$

   (b) $\left\langle \frac{11n-2}{n+3} \right\rangle \to 11$

   (c) $\left\langle \frac{\sin n}{\sqrt{n}} \right\rangle \to 0$ [S]

In[∘]:=
```
a[n_] := 6 / 7^n
TableForm[Table[{n, N[alpha[a, 0, n]]}, {n, 1, 5}]]
```

Out[∘]//TableForm=

1	1.
2	1.
3	1.
4	1.
5	1.

In[∘]:=
```
b[n_] := (11 n - 2) / (n + 3)
TableForm[Table[{n, N[alpha[b, 11, n]]}, {n, 1, 20}]]
```

Out[∘]//TableForm=

1	0.817059492511287
2	0.845487952921921
3	0.866239401436038
4	0.882062512353017
5	0.89453048200593
6	0.904610035450649
7	0.912928473834244
8	0.919910825949532
9	0.925855387800972
10	0.930977727252337
11	0.935437527770928
12	0.939355608834088
13	0.942825078777703
14	0.94591885320106
15	0.948694850270578
16	0.951199659760478
17	0.953471184752409
18	0.955540576763956
19	0.957433675458363
20	0.959172094900998

```
c[n_] := Sin[n] / Sqrt[n]
TableForm[Table[{n, N[alpha[c, 0, n]]}, {n, 1, 5}]] (* Not enlightening! *)
```

Out[∘]//TableForm=

1	7.678022798919
2	-0.743370597270885
3	0.0814885990292491
4	-10.5824991447101
5	-0.587407869808287

9. Use a Taylor polynomial to find the rate of convergence of

$$\lim_{h \to 0} \frac{\sin(h) - e^h + 1}{h} = 0.$$

12. Show that

$$(\sin h)(1 - \cos h) = 0 + O(h^3).$$

19. Use the rules of thumb for order of convergence to approximate the number of iterations it will take to achieve 12 significant digits of accuracy of π for each order of convergence. Assume each sequence starts with one significant digit of accuracy.

(a) $\alpha = 1,\ \lambda = 0.8$
(b) $\alpha = 1,\ \lambda = 0.5$ [S]
(c) $\alpha = 1,\ \lambda = 0.1$
(d) $\alpha = 1.5$
(e) $\alpha = 2$ [A]

When will d(n)=12? If α=1, then it all comes down to the log term: The formula is actually

Significant digits of accuracy: For a sequence $\langle p_n \rangle$ that converges to p with order α, the numbers of significant digits of accuracy of consecutive terms are related by the approximation

$$d(p_{n+1}) \approx \alpha d(p_n) - \log\left(\lambda |p|^{\alpha-1}\right)$$

for large enough n. In closed form (for $\alpha \neq 1$)

$$d(p_{n+k}) = (d_n - C)\alpha^k + C$$

where $C = \dfrac{\log\left(\lambda |p|^{\alpha-1}\right)}{\alpha - 1}$.

So for α=1, the rule of thumb is that $d(p_n) = d_0 - n(\log[\lambda \mid p \mid^{\alpha-1}])$

In[•]:=
```
d[n_, lambda_] := 1 - n Log10[lambda ]
Solve[d[n, 0.8] == 12, n]
Solve[d[n, 0.5] == 12, n]
Solve[d[n, 0.1] == 12, n]
```

Out[•]= {{n → 113.507362743678}}

Out[•]= {{n → 36.541209043761}}

Out[•]= {{n → 11.}}

```
d[n_, p_, lambda_, alpha_] :=
 Module[{c = Log10[lambda Abs[p] ^ (alpha - 1)] / (alpha - 1)},
  (1 - c) alpha ^ n + c
 ]
lambda = .5; (* To do it right, we'd need a lambda. One is
 not provided. That means they're using a separate rule of thumb,
one in which we simply say that the number of digits will
 go up by a power of alpha at each iteration. Hence *)

d[alpha_, digits_] := N[Log10[digits] / Log10[alpha]]
d[1.5, 12]
d[2, 12]
d[3, 13]
```

Out[•]= $\left\{\left\{n \to \frac{2\, i\, \pi\, c_1}{\text{Log}[2]} + \frac{\text{Log}[12]}{\text{Log}[2]} \text{ if } c_1 \in \mathbb{Z}\right\}\right\}$

Out[•]= {{n → 6.12853387405436}}

Out[•]= $\left\{\left\{n \to \frac{2\, i\, \pi\, c_1}{\text{Log}[3]} + \frac{\text{Log}[12]}{\text{Log}[3]} \text{ if } c_1 \in \mathbb{Z}\right\}\right\}$

Out[•]= 6.12853387405436

Out[•]= 3.58496250072116

Out[•]= 2.33471751947279