MAT225 Section Summary: 2.3
Characterizations of Invertible Matrices
Summary

Theorem 8: The Invertible Matrix Theorem

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to the identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an n x n matrix C such that $CA = I$.
11. There is an n x n matrix D such that $AD = I$.
12. A^T is invertible.

As the author says, "the power of the Invertible Matrix Theorem lies in the connections it provides between so many important concepts...."

#5, p. 132

#11, p. 132

#15, p. 132

#17, p. 133

#18, p. 133

#27, p. 133

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} S(T(\mathbf{x})) &= \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \\ T(S(\mathbf{x})) &= \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \end{aligned} \tag{1}$$

Theorem 9: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying (1).

#38, p. 133

Definition: A matrix that is nearly – but not quite – singular is said to be **ill-conditioned**. A matrix that is ill-conditioned causes trouble when the time comes to invert, and for other calculations. The **condition number** of a matrix measures how poorly conditioned a matrix is. The identity matrix has a condition number of 1, and is perfectly well-conditioned. The larger the condition number is, the closer a matrix is to singular (the condition number is infinite for a singular matrix). For a 2 x 2 matrix, the closer the determinant is to zero, the larger the condition number.

#42, p. 134