MAT225 Section Summary: 1.8
Introduction to Linear Transformations
Summary

Definition: transformation: a transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is the **domain** of T, and $\mathbb{R}^m$ is the **codomain**.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ of vectors $\mathbf{x}$ from the domain is called the **range** of the transformation T.
A transformation T is **linear** if it satisfies

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T
- $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and all scalars c.

The matrix product $A\mathbf{x}$ represents a linear transformation, as we have seen. If A is an m x n matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

More generally, a linear transformation satisfies

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p)$$

also known as the **principle of superposition**.

In this section, several important examples of linear transformation representable by matrices are given, corresponding to

- projections (Example 2),
- shears (Example 3),
- scalings (Example 4 - contractions and dilations), and

- rotations (Example 5).

As you can well imagine, these sorts of transformations are very useful to the computer scientist, among others: if you want to simulate motion in a computer game, for example, you will be constantly projecting, rotating, and scaling objects. But for translations, computer scientists have need of **affine** transformations, as described in your homework problem #30, p. 81. Have fun!