MAT225 Section Summary: 1.5
Solution Sets of Linear Systems

1. **Definitions**

   - **homogeneous system**
     a system of linear equations of the form $A_{m \text{ x n}}\mathbf{x} = \mathbf{0}_{m \text{ x } 1}$. This system always has at least one solution: the $\mathbf{0}_{n \text{ x } 1}$ vector, called the **trivial solution**. Other solutions are called **nontrivial solutions**.

2. **Theorems/Formulas**

   The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system of equations has at least one free variable.

   Theorem 6: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given vector $\mathbf{b}$, and let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

   Proof: #25, p. 56

3. Properties/Tricks/Hints/Etc.

   Observe that
   $$A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = \mathbf{b}$$
   This shows the most general form of the solution of the matrix system $A\mathbf{x} = \mathbf{b}$.

   Writing a solution set (of a consistent system) in parametric vector form:

   (a) Row reduce the augmented matrix to reduced echelon form.

(b) Express each basic variable in terms of any free variables appearing in an equation.

(c) Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.

(d) Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Example: #8, p. 55

4. Summary

This section shows us how to think of the solution set of a linear system geometrically, in terms of vectors. The main trick is to find the solution of a related system, the homogeneous system, and then find a particular solution to the system.

The solutions are some sorts of parametric representations of points (if only the trivial solution exists), lines, planes, etc.

You might relate the solutions of these equations to your history from calculus as follows:

$$[a_{11} a_{12} a_{13}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

is the same as

$$\langle a_{11}, a_{12}, a_{13} \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$$

It says that the row vector (which we might call $\mathbf{A}_1$) is perpendicular, or orthogonal, to the solution vector $\mathbf{x}$.

Then

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the same as

$$\langle a_{11}, a_{12}, a_{13} \rangle \cdot \langle x_1, x_2, x_3 \rangle = 0$$

and

$$\langle a_{21}, a_{22}, a_{23}\rangle \cdot \langle x_1, x_2, x_3\rangle = 0$$

i.e., that the $\mathbf{x}$ is orthogonal to both row vector ($\mathbf{A}_1$ and $\mathbf{A}_2$).

Now if

$$[a_{11} a_{12} a_{13}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [b]$$

this says that

$$\langle a_{11}, a_{12}, a_{13}\rangle \cdot \langle x_1, x_2, x_3\rangle = b.$$

That is, that the projection of $\mathbf{x}$ onto $\mathbf{A}_1$ is equal to b

You remember what this means: that

$$\mathbf{A}_1 \cdot \mathbf{x} = |\mathbf{A}_1||\mathbf{x}|\cos(\theta)$$

where θ is the angle between the vectors. Hence

$$A\mathbf{x} = \mathbf{b}$$

says: "the projections of $\mathbf{x}$ onto the rows of A make up the components of $\mathbf{b}$", and if

$$A\mathbf{x} = \mathbf{0}$$

then $\mathbf{x}$ is orthogonal to every row of A; or, alternatively

"$\mathbf{x}$ is orthogonal to the span of the row vectors of A".

The bang is still this:

the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.