MAT225 Test 3 (Spring 2008): Sections 4.2 through Section 5.1

Name:

Directions: Problems are equally weighted (10 points each). Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1 (10 pts): Consider the matrix $A = \begin{bmatrix} 1 & -2 & 3 & -1 & 5 \\ 2 & -2 & 1 & 1 & -5 \\ 1 & 0 & 2 & 0 & 0 \end{bmatrix}$

a. (4 pts) Find a basis for $\mathrm{Col}(A)$.

b. (4 pts) Find an explicit description of $\mathrm{Nul}(A)$.

c. (2 pts) Tell all you can about the transformation T given by $T(\mathbf{x}) = A\mathbf{x}$.

Problem 2 (10 pts): Consider the set F of real-valued functions

$$\{1 - x + x^2 - x^3, 1 + x + x^2 + x^3, x + x^3, 1 + x^2\},$$

vectors in the vector space P_3 of cubic polynomials.

a. (4 pts) What are the coordinates of this set of vectors in the standard basis ($\{1, x, x^2, x^3\}$)? (Show me by constructing the matrix M whose columns are the coordinate vectors, given in the same order as the set of functions in F.)

b. (3 pts) The span of F is clearly a subspace of P_3. Is F a basis for P_3? If not, what is the dimension of the subspace?

c. (3 pts) What is the rank of M? How is M related to the Span(F)? Illustrate by writing M as a sum of outer-products.

Problem 3 (10 pts): Consider the following vectors:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \qquad \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \text{where} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

a. (6 pts) Find the coordinate vector $[\mathbf{x}]_B$ relative to the basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

b. (2 pts) Write $\mathbf{x}$ as a linear combination of vectors in the basis B; write $\mathbf{b}_1$ as a linear combination of vectors in the basis B.

c. (2 pts) What are the two fundamental properties of a basis?

Problem 4 (10 pts): Consider the set S of matrices of the form

$$M = \begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix},$$

where a and b are real numbers.

a. (4 pts) Demonstrate that S is a vector space, a subspace of the vector space of 2 x 2 matrices with real coefficients.

b. (4 pts) Find a basis for S.

c. (2 pts) Is the matrix $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ an element of S? If so, find the coordinates of A in terms of your basis.

Problem 5 (10 pts): Given a non-zero matrix $A_{5 \text{ x } 7}$.

a. (2 pts) The null space Nul(A) is a subspace of what space? What is the maximum possible dimension of Nul(A)?

b. (2 pts) What connection, if any, exists between the dimensions of Nul(A) and Col(A)?

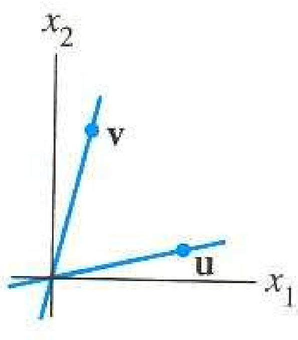
c. (2 pts) What is the maximum possible dimension of Row(A)? What connection, if any, exists between the dimensions of Nul(A) and Row(A)?

d. (4 pts) Suppose

$$A = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 9 & 7 & 5 & 3 & 1 & -1 & -3 \end{bmatrix}$$

Give bases for Row(A) and Col(A). Determine the dimension of Nul(A), and give one vector in Nul(A).

Problem 6 (10 pts): Let **u** and **v** be the vectors shown in the figure, and suppose **u** and **v** are eigenvectors of a 2 x 2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u}+\mathbf{v}$. On the same coordinate system carefully[1] plot the vectors $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$.



Extra Credit (3 pts): Let B be the basis $\{\mathbf{u}, \mathbf{v}\}$ of $\mathbb{R}^2$. What would be the form of the matrix B of this transformation T if each vector $\mathbf{x}$ were given in the coordinates of $[\mathbf{x}]_B$?

[1] I'm serious! Do a nice job: you'll lose points if you don't.