Section 5.3: Diagonalization

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Definition: diagonalizable : A square matrix A is diagonalizable if A is similar to a diagonal matrix. That is, if $A = PDP^{-1}$ for some diagonal matrix D.

Definition: The Diagonalization Theorem : $A_{n \text{ x } n}$ is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover, $A = PDP^{-1}$ (where D is diagonal) if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are the eigenvalues.

Example: #2, p. 325 Let $A = PDP^{-1}$ + compute A^4

$$A^4 = (PDP^{-1})^4 = PD^4P^{-1}$$

$$\left(A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{P^{-1}P}_{\text{collapses}}DP^{-1} = PD^2P^{-1} \right)$$

$$A^4 = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix} \left(\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \right) = \dots\dots$$

Rewrite the equation $A = PDP^{-1}$ in the form $AP = PD$ to understand what is going on: this is just the eigenvalue equation in partitioned form:

$$A[\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n] = [\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2\ \dots\ \lambda_n\mathbf{v}_n] = [\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n]D$$

where D is the diagonal matrix containing the eigenvalues.

Theorem 6: An n x n matrix with n distinct eigenvalues is diagonalizable.

Example: #10, p. 326

$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

① find eigenvalues

$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$

$= (2-\lambda)(1-\lambda) - 12 \quad = \lambda^2 - 3\lambda + 2 - 12$

$= \lambda^2 - 3\lambda + 10 \quad = (\lambda - 5)(\lambda + 2)$

$\lambda \in \{5, -2\}$

$\lambda = 5 : A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \qquad \underline{v}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2 : A + 2I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \qquad \underline{v}_{-2} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$AP = PD : \quad A \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

Theorem 7: Let A be an n x n matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

(a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

(b) The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n.

(c) If A is diagonalizable, and B_k is a basis for the eigenspace corresponding to λ_k, then the collection of the bases $B_1, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

problem matrix: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (try to find two eigenvectors)

Example: #33, p. 326