Section 5.1: Eigenvalues and Eigenvectors

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Abstract

Since linear transformations generally represent deformations of a space, it seems like it would be rather odd to find that $T(\mathbf{x})$ is just a scalar multiple of $\mathbf{x}$. That seems a rather special property.

Here we're considering the transformation $T : \mathbf{x} \mapsto A\mathbf{x}$ for $A_{n \text{ x } n}$. Eigenvectors provide the ideal basis for $\mathbb{R}^n$ when considering this transformation. Their images under the transformation are simply scalar multiples of themselves, and the multiple value is an eigenvalue.

Definition: Eigenstuff An **eigenvector** of $A_{n \text{ x } n}$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The scalar λ is called the **eigenvalue** of A corresponding to $\mathbf{x}$. There may be several eigenvectors corresponding to a given λ.

The idea is that an eigenvector is simply scaled by the transformation, so the actions of a transformation are easily understood for eigenvectors. If we could write a vector as a linear combination of eigenvectors, then it would be easy to calculate its image: if there are n eigenvectors $\mathbf{v}_i$, with n eigenvalues λ_i, then if

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \qquad \underline{u} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}_{EB}$$

then

$$A\mathbf{u} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots + c_n\lambda_n\mathbf{v}_n \qquad A\underline{u} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix}_{EB}$$

Nice, no?

If λ is an eigenvalue of matrix A corresponding to eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\Downarrow$$

This means the

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \quad \Rightarrow \quad A\underline{v} - \lambda I\underline{v} = 0$$

which is equivalent to

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

$\underline{v}$ is squished by $A - \lambda I$

So $\mathbf{v}$ is in the null space of $A - \lambda I$. If the null space is trivial, then $\mathbf{v}$ is the zero vector, and λ is not an eigenvalue. Alternatively, all vectors in the null space are eigenvectors corresponding to the eigenvalue λ (together they generate the **eigenspace** of A corresponding to λ).

As for determining the eigenvectors and eigenvalues, there is some cases in which this is extremely easy:

The eigenvalues of a diagonal matrix are the entries on its diagonal. More generally,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$

Example: #2, p. 308

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \qquad \lambda = -2 \ ?$$

$$A - (-2)I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}$$

Is rank 1 $\Rightarrow$ Null space (of interest) $-$ EB: $\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

($\exists$ ∞ # eigenvectors)

Example: #5, p. 308

Is $\underbrace{\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}}_{= \underline{x}}$ an eigenvector of $\underbrace{\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}}_{= A}$?

$A\underline{x} = \lambda \underline{x}$?

$$A\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \underline{x}$$

Yes, & the Eigenvalue $= 0$

Example: #9, p. 308

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \qquad \lambda = 1, 5$$

Find "eigenbases"

$$A - 1 \cdot I = \begin{bmatrix} 5 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \underline{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A - 5 \cdot I = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \end{bmatrix} \qquad \underline{v}_5 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Theorem 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2: If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an n x n matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

The eigenvectors and difference equations portion of this section can be illustrated with the example of the Fibonacci numbers transformation: recall that the Fibonacci numbers are those obtained by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

and $F_0 = 1$ and $F_1 = 1$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_n = \mathbf{x}_{n+1}$$

where

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

0
1 $\quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
1
2 $\quad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
3
5 $\quad x_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
8
13 $\quad x_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
21
34

The eigenvalues of this matrix are approximately $\gamma = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$ and -0.618033988749894. γ is the so-called "golden mean", which is a nearly sacred number in nature, well approximated by the ratio of consecutive Fibonacci numbers.

An eigenvector corresponding to the golden mean (normalized to have a norm of 1) is approximately

$$\begin{bmatrix} 0.5257311121191337 \\ 0.8506508083520401 \end{bmatrix}$$

so that

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5257311121191337 \\ 0.8506508083520401 \end{bmatrix} = \gamma \begin{bmatrix} 0.5257311121191337 \\ 0.8506508083520401 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$$\Rightarrow \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \gamma \end{bmatrix} = \begin{bmatrix} \gamma \\ 1+\gamma \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ \frac{1}{\gamma}+1 \end{bmatrix}$$

$$\frac{1+\gamma}{\gamma} \stackrel{?}{=} \gamma$$

$$1 + \gamma = \gamma^2$$

$$r^2 - r - 1 = 0$$