Question for today: Is $\mathbb{R}^n$ the only kind of vector space?

Answer for today: No.

Ok, but why should we care about a question like this? If $\mathbb{R}^n$ seems to work pretty well for most applications, why should we bother generalizing the ideas we have already been working with? For (at least) the following reasons:

1. We still haven't really delved into the depths of what we can learn about $\mathbb{R}^n$. We will learn more about its structure in this chapter.

2. It turns out that in many applications (engineering, physics, graphics, statistics, life sciences, etc) it helps to have a good understanding of how more general vector spaces behave.

3. The concept of a general vector space is fundamental to much of mathematical theory.

1. Examples of Vector Spaces

1. $\mathbb{R}^n$. We know it, we love it, and it's very important in many ways. Even as we discuss general vector spaces we will continually be referring back to $\mathbb{R}^n$, and we can often use $\mathbb{R}^n$ to help us 'picture' what is going on in more general settings.

2. The space of all directed arrows. Two arrows are equal if the have the same length and point in the same direction. We add them using the parallelogram rule. Given an arrow $\underline{v}$ and a real number c, $c\underline{v}$ is a new arrow whose length has been scaled by $|c|$, and if $c < 0$ then the arrow has reversed its direction.

3. The space of all doubly–infinite sequences of numbers. That is, sequences

$$\{y_i\} = (..., y_{-2}, y_{-1}, y_0, y_1, y_2, ...)$$

Given another sequence $\{z_i\}$, we add them as follows:

$$\{y_i\} + \{z_i\} = (..., (y_{-1} + z_{-1}), (y_0 + z_0), (y_1 + z_1), ...) = \{(y_i + z_i)\}$$

Also, given a real number c, we scale a sequence by writing

$$c\{y_i\} = \{cy_i\}$$

So once again, we have a collection of objects that we can add together and scale by real numbers...

4. The space of polynomials of degree $\leq n$. That is, polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$$

Given another such polynomial $q(t) = b_0 + b_1 t + ... + b_n t^n$, we can add them together. The result is still a polynomial whose degree is $\leq n$.

$$p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + ... + (a_n + b_n)t^n$$

We can also scale such polynomials using real numbers.

2. The Formal Definition

There are 10 axioms (or rules) we use to formally define a vector space.

Definition A vector space V is a collection of objects called *vectors* on which are defined two operations called *addition* and *sclar multiplication*. In addition, the following rules must hold for all vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ in V, and for all scalars c and d in $\mathbb{R}$.

(1) $\underline{u} + \underline{v}$ must be an element of V.

(2) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$

(3) $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

(4) There is a zero vector $\underline{0}$ in V such that $\underline{u} + \underline{0} = \underline{u}$.

(5) For each $\underline{u}$ in V there exists $-\underline{u}$ in V such that $\underline{u} + (-\underline{u}) = \underline{0}$.

(6) The scalar multiple of $\underline{u}$ by c, denoted by $c\underline{u}$ is an element of V.

(7) $c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$.

(8) $(c + d)\underline{u} = c\underline{u} + d\underline{u}$.

(9) 9. $c(d\underline{u}) = (cd)\underline{u}$.

(10) $1\underline{u} = \underline{u}$.

Now that we have a formal definition we are able to tell when a certain collection forms a vector space and when it does not. Let's check one of the above examples...

In example 4, we already showed that rule (1) holds. It is clear that rules (2)-(3) hold. What about (4)? Sure, define $\underline{0}$ to be the zero polynomial. The rest of the rules are also easy to check.

Non-Example 1
What if we only allow polynomials of degree 3? Is this still a vector space under the same definitions for addition and scalar multiplication? Suppose we take

$$p(t) = 5t^3 - 2t^2 + 9t + 1$$

and

$$q(t) = -5t^3 - 3t^2 + t + 2$$

Then we see that

$$p(t) + q(t) = -5t^2 + 10t + 3$$

But this is no longer of degree 3! Since the set of degree 3 polynomials is not 'closed' under addition, it is *not* a vector space.

Non-Example 2
What about the set of all polynomials with odd coefficients. Is this a vector space under the usual addition and scalar multiplication? Well,

$p(t) = 3t^2 + 1$

consider

$$p(t) = 3t^900 - 5t^32 + 7$$

This has odd coefficients, so it is an element of our collection. But if we scale it by 2, we get

$$2p(t) = 6t^900 - 10t^32 + 14$$

This is no longer in our set! Since the set of polynomials with odd coefficients is not 'closed' under scalar multiplication, it is *not* a vector space.

3. Vector Subspaces

Let's take another look at the vector space of polynomials whose degree is $\leq n$. Suppose that $n = 7$ and call this vector space V_7. We can all agree that the set of polynomials of degree ≤ 5 is a subset of V_7. Call this second (smaller) set V_5.

What do we know about the set V_5? For one thing, its a vector space itself! So we have a smaller vector space sitting inside a larger vector space...this turns out to be an important concept.

4. Formal Definition of Vector Subspaces

Given a subset of a vector space, how can we check that the subset is itself a vector space?

Definition
A subspace is a subset H of V that has three properties:

(1) The zero vector of V is in H.

(2) H is closed under vector addition. That is, given any vectors $\underline{u}$ and $\underline{v}$ from H, $\underline{u} + \underline{v}$ is also in H.

(3) H is closed under scalar multiplication. That is, given any vector $\underline{u}$ in H and any scalar c, $c\underline{u}$ is also in H.

This definition is simply ensuring that H is itself a vector space. Since H sits inside of V, most of the properties from the definition of a vector space are automatically satisfied. We only need to check the three listed above.

Example 1
Every vector space is a subspace of itself.

Example 2
We already saw that polynomials of degree ≤ 5 form a subspace of polynomials of degree ≤ 7.

Example 3?
Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

Nope. Vectors in $\mathbb{R}^2$ have two entries while vectors in $\mathbb{R}^3$ have three. Therefore, $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. However, consider the following:

$$H = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = span\{\underline{e}_1, \underline{e}_2\}$$

Is this a subspace of $\mathbb{R}^3$? Well, it is definitely a subset of $\mathbb{R}^3$. So now we need to check the three properties from the definition.

The zero vector is certainly in H. That is, $\underline{0} = 0\underline{e}_1 + 0\underline{e}_2$.

If we add or scale vectors in H, the third entry is still zero, so the linear combination is still in H. That is, H closed under vector addition and scalar multiplication.

Therefore, H is indeed a subspace of $\mathbb{R}^3$. In addition, we notice that H "looks" a lot like $\mathbb{R}^2$...We can picture H as a plane in $\mathbb{R}^3$ that passes through the origin.

Example 4
What about a plane P in $\mathbb{R}^3$ that does *not* pass through the origin? Will P still be a subspace? No, because it fails the first property in the definition of a subspace. That is, P is not a vector space, so it cannot be a subspace.

5. Subspaces Spanned by a Set

Let's think about Example 3 above. We defined H as the span of two vectors in a larger vector space, and we ended up with a subspace. Will this always work?

Given two vectors $\underline{u}_1$ and $\underline{u}_2$ in V, let's show that $H = span\{\underline{u}_1, \underline{u}_2\}$ is a subspace of V.

We first check that $\underline{0}$ is in H. Then we check the other two properties...

In general, we get the following theorem:

Theorem
If $\underline{v}_1$, $\underline{v}_2$,...,$\underline{v}_p$ are vectors in a vector space V, then $span\{\underline{v}_1, ..., \underline{v}_p\}$ is a subspace of V.

We can use this to check that $H = \{(a-3b, b-a, a, b)$: a,b in $\mathbb{R}\}$ is a subspace of $\mathbb{R}^4$. (write it as the space of two vectors!)

$(a-3b, b-a, a, b) =$

$a(1, -1, 1, 0) +$
$b(-3, 1, 0, 1)$

HW: Section 4.1 #2, 4, 5, 6, 7, 8, 12, 14, 26, 30