Section 2.3: Characterizations of Invertible Matrices

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Abstract

Theorem: 8 : The Invertible Matrix Theorem

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is invertible.

(b) A is row equivalent to the identity matrix. $[A\ I_n] \sim [I_n\ A^{-1}]$

(c) A has n pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

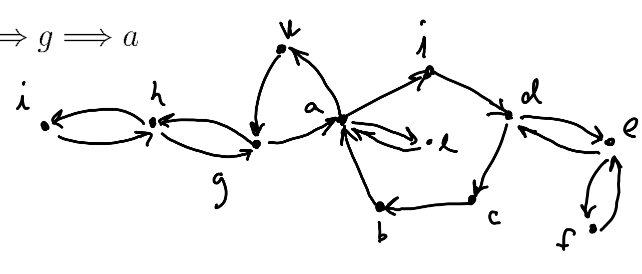
(j) There is an n x n matrix C such that $CA = I$.

(k) There is an n x n matrix D such that $AD = I$.

(l) A^T is invertible.

The proof is interesting:

- The first piece is $a \Longrightarrow j \Longrightarrow d \Longrightarrow c \Longrightarrow b \Longrightarrow a$
- Then $a \Longrightarrow k \Longrightarrow g \Longrightarrow a$



- $g \iff h \iff i$
- $d \iff e \iff f$
- $a \iff l$

As the author says, "the power of the Invertible Matrix Theorem lies in the connections it provides between so many important concepts...."

Example: #5, p. 132

$$\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \not\sim I_3$$

(one step of row reduction shows that)

Example: #11, p. 132

a. $A\underline{x} = \underline{0}$ has only trivial soln $\Rightarrow A \sim I_n$?
Yes.

b. Columns of A span $\mathbb{R}^n \Rightarrow$ columns linearly indep?
Yes

c. False; A must be invertible

d. $A\underline{x} = \underline{0}$ has a non-trivial soln $\Rightarrow$ A has $< n$ pivots ?
$\Rightarrow$ dependence in columns
$\Rightarrow < n$ pivot positions

e. A^T not invertible $\Rightarrow$ A not invertible. Yes.

Example: #15, p. 132

No: dependence in columns $\Rightarrow$ fewer than n pivots.

Example: #17, p. 133

If A invertible, then columns of A^{-1} are linearly independent - Yes: if A invertible, (A^{-1}) is invertible - so its columns are linearly independent.

Example: #18, p. 133

$C_{6\times 6}$ $\quad C\underline{x} = \underline{v}$ consistent for every $v \in \mathbb{R}^6$

$\Rightarrow C$ is invertible (map is onto)

$\Rightarrow$ every solution is unique

Example: #27, p. 133

AB invertible $\Rightarrow A$ invertible

$\exists W \,/\, (AB)W = I$ (defn) or IMT

$\therefore A(BW) = I \Rightarrow A$ invertible

Definition: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} S(T(\mathbf{x})) &= \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \\ T(S(\mathbf{x})) &= \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \end{aligned} \tag{1}$$

Theorem: 9 : Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T (see section 1.9, p. 83 for more – basically the columns of A are the images $T(\mathbf{e}_j)$ of the columns $\mathbf{e}_j$ of the identity matrix). Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying (1).

$T : \underline{x} \to A\underline{x}$

$S : \underline{x} \to A^{-1}\underline{x}$

Example: #38, p. 133

$T : \mathbb{R}^n \to \mathbb{R}^n$

$T(\underline{u}) = T(\underline{v})$ for $\underline{u} \neq \underline{v}$

$A\underline{x} = T(\underline{u})$ has how many solutions? ∞

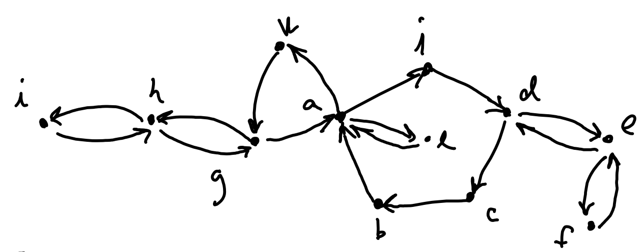
$A\underline{u} = T(\underline{u})$

$A\underline{v} = T(\underline{u}) = T(\underline{v})$

No, T can't map $\mathbb{R}^n$ onto $\mathbb{R}^n$ - we've lost a dimension

Definition: : A matrix that is nearly – but not quite – singular is said to be **ill-conditioned**. A matrix that is ill-conditioned causes trouble when the time comes to invert, and for other calculations. The **condition number** of a matrix measures how poorly conditioned a matrix is. The identity matrix has a condition number of 1, and is perfectly well-conditioned. The larger the condition number is, the closer a matrix is to singular (the condition number is infinite for a singular matrix). For a 2 x 2 matrix, the closer the determinant is to zero, the larger the condition number.

Example: #42, p. 134



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \end{array}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$