Question for today: How can two or more matrices relate to one another?

Examples:

1. If we have a 3-D model and we want to rotate it five times in succession, we can use five matrices, one for each rotation. But can we somehow combine these rotations into a single rotation? What happens to the matrices when we do this?

2. Suppose we want to analyze different aspects of a single matrix. Sometimes we can do this by breaking it down into two different matrices that are related to one another via an operation we will discuss today.

1. Notation

Given a matrix A, we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & & a_{1n} \\ a_{21} & \ddots & & & \\ \vdots & & a_{ij} & & \\ & & & \ddots & \\ a_{m1} & \dots & & & a_{mn} \end{bmatrix} = [a_{ij}] = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix}$$

We say that $A = B$ if they are the same size and their corresponding entries (or columns) are the same. For example, if we have matrices A, B, and C given by

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$$

then we say that $A = B$ but $A \neq C$.

2. Sums and Scalar Multiples

Suppose that we have two 2×3 matrices A and B given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Suppose that we have a vector $\underline{x}$ and we want to look at $A\underline{x} + B\underline{x}$. Instead of multiplying $\underline{x}$ by two different matrices and then adding the results, could we find a single matrix C so that $A\underline{x} + B\underline{x} = C\underline{x}$?

Any ideas??? $A\underline{x} + B\underline{x} = (A+B)\underline{x}$ what is this?

Well, instead of adding $A\underline{x}$ and $B\underline{x}$ we can just add A and B to get C. But what does it mean to add matrices?? How would you define matrix addition?? What makes the most sense?? Remember, we want $A\underline{x} + B\underline{x} = C\underline{x}$. Let's see how this breaks down...

Write $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then we have

$$A\underline{x} + B\underline{x} = \begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix} + \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ 5x_1 + 6x_2 + 7x_3 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2)x_1 + (2+3)x_2 + (3+4)x_3 \\ (4+5)x_1 + (5+6)x_2 + (6+7)x_3 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2) & (2+3) & (3+4) \\ (4+5) & (5+6) & (6+7) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2) & (2+3) & (3+4) \\ (4+5) & (5+6) & (6+7) \end{bmatrix} \underline{x}$$

So what's the logical choice for C? We let $C = \begin{bmatrix} (1+2) & (2+3) & (3+4) \\ (4+5) & (5+6) & (6+7) \end{bmatrix}$.

Ok, so what have we done? We've defined what it means to add matrix A to matrix B. To get $A + B$, we simply add the corresponding entries of A and B together.

Definition of Matrix Addition:
If $A = [a_{ij}]$ and $B = [b_{ij}]$, we have $A + B = [(a_{ij} + b_{ij})]$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}$$

Using this kind of thinking, what happens when we scale a matrix by a number r? That is, what is rA?

Definition of Scalar Multiplication:
If $A = [a_{ij}]$ and r is a real number, then $rA = [(ra_{ij})]$.

Now that we have some definitions, let's try it out using A and B as below:

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

What is $A - 2B$?

$$= \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

Properties of Addition and Scalar Multiplication:
Let A, B, and C be $m \times n$ matrices and let r, s be any real numbers. Then we have
a) $A + B = B + A$ commutative
b) $(A + B) + C = A + (B + C)$
c) $A + [0] = A$
d) $r(A + B) = rA + rB$
e) $(r + s)A = rA + sA$
f) $r(sA) = (rs)A$

To check these properties all we have to do is compare the corresponding entries in the matrices on either side of the = signs.

3. Matrix Multiplication

Here is the fun part! Given an $m \times n$ matrix A and a $n \times p$ matrix B, consider the following situation:

$$\begin{tikzcd} \underline{x} \arrow[r, "B"] \arrow[rr, bend right, "?\ C\underline{x}\ ?"'] & B\underline{x} \arrow[r, "A"] & A(B\underline{x}) \end{tikzcd}$$

Is there a single matrix that will take us from $\underline{x}$ to $A(B\underline{x})$? Well, yes, there is. We call it AB. For examples, consider the following matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Now let $\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We can see that, indeed, $\underline{A(B\underline{x}) = C\underline{x}}$. Therefore, $C = AB$.

$$B\underline{x} = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$A(B\underline{x}) = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 13 \\ 2 \end{bmatrix} = \begin{bmatrix} 32 \\ 3 \end{bmatrix}$$

$$C\underline{x} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 32 \\ 3 \end{bmatrix}$$

This is what it means for "$AB = C$"

ie C sends vectors to the same place as does B followed by A

To actually get a formula for AB, let's do what we did when we defined addition. Suppose $B = [\ \underline{b}_1 \ \ \dots \ \ \underline{b}_n\ ]$. Then

$$A(B\underline{x}) = A(x_1\underline{b}_1 + \dots + x_n\underline{b}_n)$$

$$= A(x_1\underline{b}_1) + A(x_2\underline{b}_2) + \dots + A(x_n\underline{b}_n)$$

$$= x_1(A\underline{b}_1) + x_2(A\underline{b}_2) + \dots + x_n(A\underline{b}_n)$$

$$= \underline{[\ A\underline{b}_1 \ \ A\underline{b}_2 \ \ \dots \ \ A\underline{b}_n\ ]}\,\underline{x}$$

This matrix is AB

So what's the formula for AB? Well, we see that $AB = [\ A\underline{b}_1 \ \ A\underline{b}_2 \ \ \dots \ \ A\underline{b}_n\ ]$.

Definition of Matrix Multiplication:
Given an $m \times n$ matrix A and an $n \times p$ matrix $B = [\ \underline{b}_1 \ \ \dots \ \ \underline{b}_p\ ]$, we have $AB = [\ A\underline{b}_1 \ \ \dots \ \ A\underline{b}_p\ ]$.

Let's try this out. Given A and B, find AB.

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

$$A\underline{b}_1 = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ -2 \end{bmatrix} \qquad A\underline{b}_2 = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

So $AB = [A\underline{b}_1 \ \ A\underline{b}_2] = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$

Let's take a closer look at how we calculated AB in the last example. Is there another way to perform these calculations that wouldn't require so much writing? Let's try calculating AB when

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

What is the 1, 1 entry of AB?

$$A\underline{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ 23 \end{bmatrix}$$

the 1,1 entry is 7

Look at how we combine the 1st row of A with the 1st column of B!

What is the 1, 2 entry of AB? Let's try combining the 1st row of A with the 2nd column of B like we did above...

$$A\underline{b}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \\ 34 \end{bmatrix}$$

So the 1, 2 entry is 10

We call this the *row-column rule* for matrix multiplication. Let's finish finding AB this way.

Now you find AB: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$(AB)_{1,1} = 8$ using the row/column rule

$$AB = \begin{bmatrix} 8 & 8 & 4 \\ 14 & 14 & 7 \end{bmatrix}$$

Can you find BA?

Finding BA doesn't make sense, does it? Why is this? Let's look back at our first definition of AB. For an $m \times n$ matrix A and an $n \times p$ matrix B we said that

$$AB = \left[\; A\underline{b}_1 \quad A\underline{b}_2 \quad \ldots \quad A\underline{b}_p \; \right]$$

What do we notice about the dimensions of A and B??

Well, for $A\underline{b}_i$ to be defined, we must have the number of columns in A equal to the number of rows in B. This is the only requirement.

So given an $m \times n$ matrix A and an $n \times p$ matrix B, AB is a well defined matrix and we know how to compute it. But what is the dimension of AB?

Answer: AB will be an $m \times p$ matrix.

Properties of Matrix Multiplication
Suppose A is $m \times n$ and that all of the following products are well defined. Then we have:
a) $A(BC) = (AB)C$
b) $A(B+C) = AB + AC$
c) $(B+C)A = BA + CA$
d) $r(AB) = (rA)B = A(rB)$
e) $I_m A = A = A I_n$

Why do we need both (b) and (c)? Aren't they saying the same thing? Isn't $A(B+C) = (B+C)A$? Hmmm...

Recall the following to matrices:

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

We already found that

$$AB = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

But let's see what BA is equal to. Is BA even defined???

So we see that $AB \neq BA$! Because of this, we say that matrix multiplication is **non-commutative**. That is, we cannot always switch from AB to BA without possibly changing the resulting matrix or even getting something that is not defined.

Check out the box on page 114 for some warnings related to the non-commutativity of matrix multiplication.

4. The Transpose of a Matrix

One last item...given a matrix A, sometimes we want to consider the new matrix obtained by taking the columns of A, turning them sideways, and making them into rows. We call this new matrix A^T. We say A^T as A *transpose*.

Definition
Given an $m \times n$ matrix A, the $n \times m$ matrix A^T is formed by taking the columns of A and turning them into rows.

Examples:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -1 & 7 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

Properties of matrix transposition
a) $(A^T)^T = A$
b) $(A+B)^T = A^T + B^T$
c) $(rA)^T = r(A^T)$
d) $(AB)^T = B^T A^T$

HOMEWORK!
p. 116 # 1, 2, 7, 9, 10, 12, 20, 21, 22, 23