Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$

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Abstract

We encounter yet another representation for a system of linear equations – will it never end?! This is the last we'll examine, and probably the most important. Theorem four pulls all these forms together: spans, pivots, linear combinations, and matrix equations collide!

"A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector." p. 40

Matrix/vector multiplication is defined. One form that I find particularly useful is the so-called "row-vector rule": a row of the matrix slams into the variable vector $\mathbf{x}$, to produce a single entry in the $\mathbf{b}$ vector.

Definition: product of matrix A and vector $\mathbf{x}$

If A is an m x n matrix, with columns $\mathbf{a}_1$, $\mathbf{a}_2 \ldots$, $\mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$ is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Example: #4, p. 47

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

We now have four ways of writing a system of equations(!), as given in

Theorem Three (p. 42): If A is an m x n matrix, with columns $\mathbf{a}_1$, $\mathbf{a}_2 \ldots$, $\mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \quad \mathbf{b}]$$

Example: #9, p. 47

$$\begin{array}{rcrcrcl} 3x_1 & + & x_2 & - & 5x_3 & = & 9 \\ & & x_2 & + & 4x_3 & = & 0 \end{array}$$

$$x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -5 & 9 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

Existence of solutions is given by the following theorem:

Theorem Four (p. 43): Let A be an m x n matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

(i) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(ii) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(iii) The columns of A span $\mathbb{R}^m$.

(iv) A has a pivot position in every row.

Example: #14, p. 48

Let $\underline{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ + $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$

Is $\underline{u}$ in the span of the columns of A?

Consider

$A\underline{x} = \underline{u}$; $\begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix}$; $\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}x_1 + \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}x_2 + \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}x_3 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$

A handy way to think about matrix multiplication: Row-Vector rule for computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the *ith* entry in the vector $A\mathbf{x}$ (yes, it's a vector!) is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example: Revisit #4, p. 47

$$\underset{2\times 3}{\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 8+3+-4 \\ 5+1+2 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 7 \\ 8 \end{bmatrix}}$$

Slam rows from matrix A into the vector $\underline{x}$, & generate components of the product.

Theorem Five (p. 45): If A is an m x n matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

(i) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

(ii) $A(c\mathbf{u}) = c(A\mathbf{u})$

Example: #35, p. 49

$A_{3\times 4}$; $\underline{y}_1, \underline{y}_2 \in \mathbb{R}^3$; $\underline{w} = \underline{y}_1 + \underline{y}_2$

Suppose $\exists \underline{x}_1$ and $\underline{x}_2$ / $A\underline{x}_1 = \underline{y}_1$ and $A\underline{x}_2 = \underline{y}_2$.

How do we know that $A\underline{x} = \underline{w}$ is consistent?

$$A\underbrace{(\underline{x}_1 + \underline{x}_2)}_{\underline{x}} = A\underline{x}_1 + A\underline{x}_2 = \underline{y}_1 + \underline{y}_2 = \underline{w}$$

#36 $A = \begin{bmatrix} \end{bmatrix}_{5\times 3}$ $\underline{y} = \begin{bmatrix} \end{bmatrix}_{3\times 1}$ $\underline{z} = \begin{bmatrix} \end{bmatrix}_{5\times 1}$

$$A\underline{y} = \underline{z}$$

Is $A\underline{x} = 4\underline{z}$ consistant?

$\underline{x} = 4\underline{y}$ is a soln:

$$A(4\underline{y}) = 4(A\underline{y}) = 4\underline{z}$$