Section 1.3: Vector Equations

January 23, 2008

Abstract

Vectors provide a wonderful way for us to write systems of equations compactly. You should already be familiar with two-d and three-d vectors from calculus classes. We now want to extend notions from those spaces into n-dimensional space. For example, vector addition is carried out component-wise.

The interesting new concept introduced in this section is that of **span**: roughly, the span of a set of vectors $\{\mathbf{v}_1, ...\mathbf{v}_p\}$ is the subspace generated by linear combinations of the vectors $\mathbf{v}_i$. The span represents the set of vectors $\mathbf{b}$ that can be solutions of the system

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

Definition: A **vector** is a matrix with only a single column ("column vector"). The entries are called the **components** of the vector.

- **zero vector**: the vector whose components are all 0: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- **one vector**: the vector whose components are all 1: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- **scalar multiple** of a vector: a product of a constant ("**scalar**") and a vector, the operation being carried out component-wise: e.g.

$$\alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{bmatrix}$$

  Note: here's a notational issue. Vectors will generally be in boldface (on the board I'll either underline them, or overline them, depending on my mood, time of day, and what I had for breakfast). The components of named vectors are generally written with the same name, only without bold/overline/underline, and with subscripts. Notice that the components $\{v_1, v_2, v_3\}$ of the vector $\mathbf{v}$ above are not at all the same as the vectors listed in the abstract, $\{\mathbf{v}_1, ...\mathbf{v}_p\}$. Components are (generally) numbers....

- **vector sum**: the vector created by adding two vectors, the sums being carried out component-wise. Naturally the vectors must have the same length....

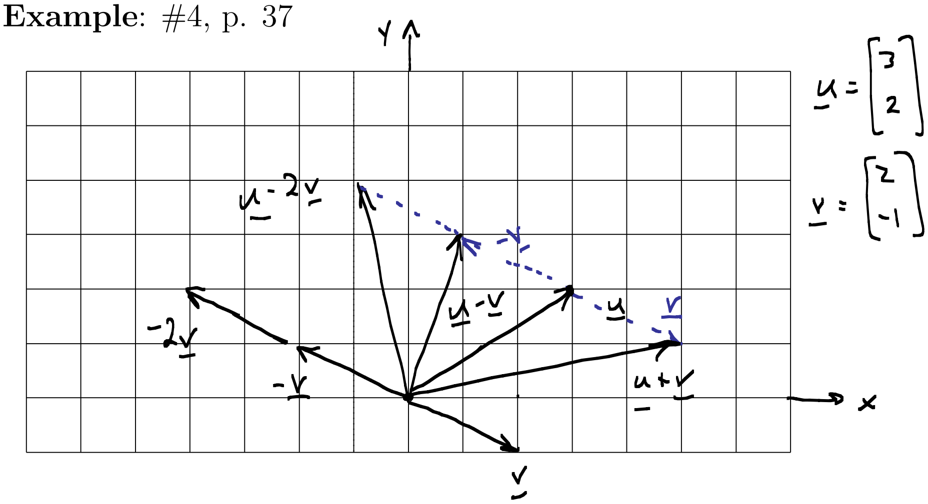
$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{bmatrix}$$

  Note: Geometrically, the sum of vectors can be found using the "parallelogram rule": the butt of vector $\mathbf{v}_2$ is placed at the tip of the vector $\mathbf{v}_1$, and the vector from the butt of $\mathbf{v}_1$ to the tip of $\mathbf{v}_2$ is the sum.

- **linear combination** of vectors: any sum of vectors scaled by coefficients. E.g.,

$$\alpha\mathbf{u}+\beta\mathbf{v}=\alpha\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\beta\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}\alpha u_1+\beta v_1\\\alpha u_2+\beta v_2\\\alpha u_3+\beta v_3\end{bmatrix}$$

  Example: #4, p. 37



- **span**: the span of a set of vectors $\{\mathbf{v}_1,...\mathbf{v}_p\}$ is the subspace generated by linear combinations of the vectors $\mathbf{v}_i$. The span represents the set of vectors $\mathbf{b}$ that can be solutions of the system

$$a_1\mathbf{v}_1+\ldots+a_p\mathbf{v}_p=\mathbf{b}$$

  Q: What is the geometry of a span? What cases should be considered?

  Varies from a zero-dimensional space (if the set is only the zero vector), to the space itself.

- The vector equation

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

has the same solution as the linear system whose augmented matrix is

$$[\mathbf{v_1} \quad \mathbf{v_2} \quad \ldots \quad \mathbf{v_p} \quad \mathbf{b}]$$

In this case, the variables – the unknowns – would be the coefficients a_i, and a solution would consist of the appropriate possibilities of values for those coefficients.

Example: #9, p. 37

$$\begin{array}{rcrcrcl} & & x_2 & + & 5x_3 & = & 0 \\ 4x_1 & + & 6x_2 & - & x_3 & = & 0 \\ -x_1 & + & 3x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 5 & 0 \\ 4 & 6 & -1 & 0 \\ -1 & 3 & -8 & 0 \end{bmatrix} \longrightarrow x_1\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: #12, p. 38. In this problem we throw you for another loop, by using the letter "a" for <u>vectors</u>! You have to pay attention, and not let us mess you up too badly just by poor notation....

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix} \text{ + look for solutions...}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix}$$

from this we can see it's inconsistent!

$$5x_2 + 4x_3 = 1$$
$$5x_2 + 4x_3 = 3$$

- Two vectors are equal only if they have the same dimensions, and their components are the same.

- Algebraic properties of the **vector space** $\mathbb{R}^n$: for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d,

  (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ commutative

  (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ associative

  (iii) $\mathbf{u} + \mathbf{0} = \mathbf{u}$ additive identity

  (iv) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ inverses (additive)

Proof of (i): $\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{bmatrix} = \underline{v} + \underline{u} \qquad \text{QED}$$

(v) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ } distributivity

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$ associativity

(viii) $1\mathbf{u} = \mathbf{u}$ multiplicative identity

Example: #21, p. 38

$\underline{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the Span$\{\underline{u}, \underline{v}\}$ for all h, k.

($\underline{u}$ & $\underline{v}$ span the whole space $\mathbb{R}^2$).

$\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$ & check for consistency.

$\sim \begin{bmatrix} 2 & 2 & h \\ 0 & 2 & k+\frac{1}{2}h \end{bmatrix}$ consistent

$\sim \begin{bmatrix} 2 & 0 & h-(k+\frac{1}{2}h) \\ 0 & 2 & k+\frac{1}{2}h \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{4}h - \frac{1}{2}k \\ 0 & 1 & \frac{1}{4}h + \frac{1}{2}k \end{bmatrix}$

Example: #27, p. 38

$\underline{v}_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ $\quad \underline{v}_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$

a) $5\underline{v}_1$? 5 days production in mine 1.

b) $x_1 \begin{bmatrix} 20 \\ 550 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 500 \end{bmatrix} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{bmatrix}$

we solve this system to obtain the solution

$$\boxed{\begin{aligned} x_1 &= 1.5 \\ x_2 &= 4 \end{aligned}} \text{ (days)}$$

#28 a. $x_1 \cdot 27.6 + x_2 \cdot 30.2$

b. $\underline{\text{Output}} = x_1 \underbrace{\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2 \begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}}_{=}$

c. $\begin{bmatrix} 162 \\ 23610 \\ 162.3 \end{bmatrix} =$