Section 1.1: Systems of Linear Equations

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Abstract

We begin with linear equations, and systems of linear equations. The variable names are essentially irrelevant to the solution set, so matrix notation eliminates the need to even give them names!

Given a system, the idea is to replace the system by one that's easier to solve, yet retains the solutions of the original system. This is done by elementary row operations (replacement, interchange, and scaling). Finally a triangular system is obtained, and the solution can be obtained by back-substitution (if a solution exists!).

Geometrically, the solution set of a system of linear equations corresponds to the intersection of linear objects embedded in space. There may be no solution, a unique solution, or an infinite number of solutions.

- **Motivation**: eHarmony and its 29 dimensions: "eHarmony is the only relationship site on the web that creates compatible matches based on 29 dimensions scientifically proven to predict happier, healthier relationships."

- Definition: **linear equation**

  an equation of the form

  $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  where b and the a_i are generally real or complex numbers, and the x_i are variables.

  Definition: **system of linear equations**

  A set of linear equations which are to be true simultaneously

  Example:

  $$\begin{array}{c} 1 \quad 2x + 4y = -4 \\ 5x + 7y = 11 \end{array}$$

  $1 \quad 2x + 4y = -4$

  $2 \quad 5x + 7y = 11 \quad \Rightarrow y = \frac{1}{7}(11-5x)$

$$2x + 4\left(\frac{1}{7}(11-5x)\right) = -4$$

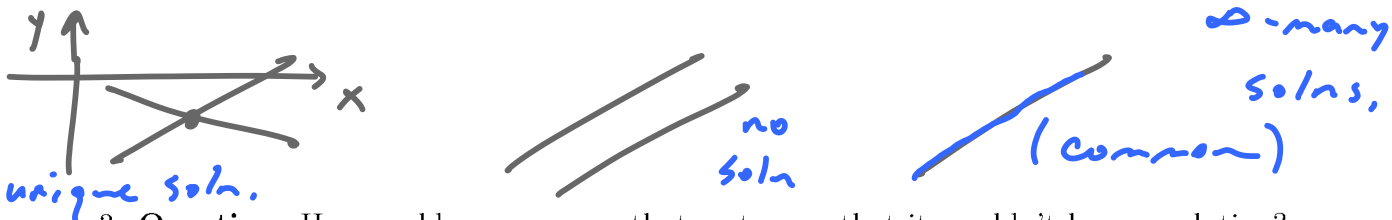
Definition: **solution set** of a linear system

the set of all solutions of a linear system

1. **Question**: How did we solve that back in High School?

   Solve ② for y as a function of x; sub back into ①, giving $x = 12$, $y = -7$ (check answer!)

2. **Question**: What are the geometric possibilities?



3. **Question**: How could we screw up that system so that it wouldn't have a solution?

   ① $2x + 4y = -4$

   ② $5x + 10y = 11$

4. **Question**: How can technology help?

   Lots faster; bigger problems; graphics

- **Theorem:**

  A system of linear equations has either

  - no solution, or
  - exactly one solution, or
  - infinitely many solutions.

  ① $2\,\square + 4\,\bigcirc = -4$

  ② $5\,\square + 7\,\bigcirc = 11$

- Definition: **matrix**

  a rectangular array of numbers (or even more general objects)

Matrices allow us to eliminate unnecessary stuff! They suppress variable names, which are irrelevant to the solution.

1. **coefficient matrix**

   matrix of the coefficients of a linear system

   $$\begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}_{2\times 2}$$

   ① $2x + 4y = -4$

   ② $5x + 7y = 11$

2. **augmented matrix**

   matrix of the coefficients of a linear system and an added column containing the right hand side of the linear system

   $$\begin{bmatrix} 2 & 4 & -4 \\ 5 & 7 & 11 \end{bmatrix}_{2\times 3}$$

3. **size of a matrix**

   indicates the number of rows and the number of columns of a matrix (e.g. 3 by 4 - 3 rows, 4 columns)

$-2.5$① $+$②

"back substitute"

$2x - 28 = -4$

$x = 12$

$$\begin{array}{c} ① \\ ② \end{array}\begin{bmatrix} 2 & 4 & -4 \\ 5 & 7 & 11 \end{bmatrix}_{2\times 3} \sim \begin{bmatrix} 2 & 4 & -4 \\ 0 & -3 & 21 \end{bmatrix} \Rightarrow -3y = 21$$

$y = -7$

- Using matrix notation, we can now solve systems using **elementary row operations**:
  - The point: Replace a given system by another system with the same solution set that's easier to solve
  - What's legal?
    1. replacement of a row by a sum of the row and multiples of other rows
    2. interchange of rows
    3. scaling of a row by a nonzero constant

  Definition: two matrices are **row-equivalent** if they can be transformed back and forth using elementary row operations.

Example: #12, p. 11

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow 0 \cdot x_3 = 3$$

inconsistent (no solution)

parallel, distinct planes.

How can technology hurt?

- Fundamental question about the solution's **Existence and Uniqueness**:
  - Is there at least one solution? (consistent; otherwise inconsistent)
  - Given that a solution exists (existence), is the solution unique (uniqueness)?

Example: #20, p. 12

Determine the values of h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{array}{c} ① \\ ② \end{array} \begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & h & -3 \\ 0 & \underbrace{4+2h} & 0 \end{bmatrix} \quad 2①+②$$

because this is zero, we have a solution for any h: $y=0$

What if $4+2h=0$? Any choice for y will work, +

$x + -2y = -3$

is a line of solutions (no uniqueness).

- **Additional examples:** p. 11, #15, 26, 32

#22

$$\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & h \\ 0 & 0 & 5+3h \end{bmatrix} \quad 3①+②$$

$$0\cdot x + 0 \cdot y = 5+3h$$