MAT129 overview: Chapters 2-6

The course can be separated into two halves:

- the differentiatial calculus (differentiation), and
- the integral calculus (integration),

The first, differentiation, is concerned with slopes, and rates of change. In the example of a car in motion, it's about the speedometer. It's about the instantaneous rate of change which the speed represents. The second, integration, is about the calculation of areas: it's about the odometer (an accumulation of distance over time, as the speed has varied).

I'm going to start backwards, with the integral, and go all the way back to the beginning. An integral is frequently thought of as the computation of an area, but I want you to remember this important formula:

$$I = \int_a^b dI(x)$$

Every integral created to determine I is a sum of tiny chunks of I, which we call $dI(x)$. At each point x, as x slides from a to b, we find a little chunk $dI(x)$, and we accumulate those. By the time we're done, we have the total I.

The chunks dI should have the units of I – even though we think of integrals as areas, they don't necessarily have units of ft^2 (for example). The units are dictated by I.

Our job in every integral is to figure out what is the best way to chop up our problem into tiny chunks of I; that usually determines the independent variable x, and the limits along which we should slide (a to b).

So, for example, in the odometer/speedometer situation, we might be looking for the total distance traveled in a car with a known speed function $s(t)$ (in mph) over the interval $t \in [0, 3]$ (in hours).

I want to accumulate the local distances traveled into a total distance traveled, so an integral like

$$D = \int_a^b dD(x)$$

seems appropriate, where I've got to figure out how to slice up the distance. Well, we're travelling for three hours, and at each moment I know my speed. Using the "dirt" formula, $d = r \cdot t$ (that is, distance equals rate times time), I figure that if I make my time intervals short enough (Δt), we can consider the speed constant over those intervals. Then the distance traveled over the i^{th} short interval (Δd_i) will just be $\Delta d_i = r(t_i) \cdot \Delta t$.

This analysis suggests everything we need to know:

a. The independent variable will be time t (rather than our favorite variable, x);

b. The limits of integration will be $a = 0$ and $b = 3$ (in hours); and

c. the form of $dD(t)$ will be played by $s(t)dt$: the product of speed, in mph, and time in hours, resulting in a quantity with units miles.

Thus

$$D = \int_0^3 s(t)dt$$

is the solution (and we merrily plug this into our calculators, unless we are lucky).

If we know an anti-derivative of the function $s(t)$ (that's what it means to be lucky!), then we can actually solve this analytically. Assume that we're moving down a straight road, and that the speed $s(t)$ along this road is the time rate of change of the position on the road, $P(t)$: if we know P, then we can write down the answer:

$$D = \int_0^3 s(t)dt = P(3) - P(0)$$

(which just says that the total displacement is the difference in positions at times 3 and 0 – that makes sense to you, I hope!).

There is difference between speed and velocity: speed is always positive, while velocity is either positive or negative (along a straight road – or it can actually be a vector quantity, if we allow for travel in two- or three-dimensional space.

Now, often we don't have actual functions $s(t)$, but rather we have readings every second or so (your car's computer may well be keeping track of your speed every second – useful in the event of a crash – your car spying on you, and then ratting you out if it was your excessive speed that caused the crash!).

In that case, we can approximate the integral, instead: we can think of $s(t)$ as being essentially constant on 1 second intervals, so we could simply write

$$D = \int_0^3 s(t)dt \approx \sum_{i=1}^{3 \cdot 3600} s(t_i) \cdot 1second \cdot \frac{1hour}{3600seconds}$$

If we could make our time steps smaller, then we would get better and truer results: in fact, if we could make the width of a time step $\Delta t = \frac{3-0}{N}$ go to zero (by letting $N \to \infty$), then

$$D = \lim_{N\to\infty} \sum_{i=1}^{N} s(t_i) \cdot \Delta t_i = \int_0^3 s(t)dt$$

If we can't do that, however, then we might reconsider our approximation

$$D_{RRR} \approx \sum_{i=1}^{3 \cdot 3600} s(t_i) \cdot 1second \cdot \frac{1hour}{3600seconds}$$

This is the right rectangle (or endpoint) rule; we could similarly consider the **left** rectangle rule,

$$D_{LRR} \approx \sum_{i=0}^{3 \cdot 3600-1} s(t_i) \cdot 1second \cdot \frac{1hour}{3600seconds}$$

Then we recall that, whenever we have two estimates, we have a third: the average, so

$$D_T \approx \frac{D_{RRR} + D_{LRR}}{2}$$

(the trapezoidal rule) is an improvement. We also considered the midpoint method.

Finally, we might want to consider this problem more generally: your odometer reading is actually a function of time: did you ever think of that? At any moment you can look at your odometer, and it will show you the result of an integral related to D:

$$ODOM(T) = \int_0^T s(t)dt$$

The fact that we're talking about speed, rather than velocity, takes care of a problem. Suppose that $s(t)$ represented the velocity. If you tool down a road at 60 mph for an hour and a half, turn around, and come back at 60 mph for an hour and a half (actually -60 mph), then your change in position is 0! You left home, and arrived back at home, so

$$D = \int_0^3 s(t)dt = P(3) - P(0) = 0$$

Yikes! But if we'd slapped absolute values on the velocity, and consider $s(t)$ the speed, then we would have gotten the correct answer (180 miles – 3 hours at 60 mph).

1 Techniques

The book is full of tricks for solving various calculus problems. For example, there is an integration technique called "substitution", which is simply the chain rule backwards. **Every differentiation rule can be turned into an integration rule,** and *vice versa.*

The general rule for solving integrals is this: **stare at the integral until the anti-derivative comes to you**. If nothing comes to you, then you should just use numerical approximation (e.g. your calculator, Mathematica).

2 Fundamental Theorem

The theorem that ties together the two branches of calculus, differentiation and integrals, assures us that

$$I = \int_a^b f(t)dt = F(b) - F(a)$$

where F is an anti-derivate of f. So **an anti-derivative solves the area problem**.

A tiny change turns Part I of the Fundamental Theorem into Part II:

$$I = \int_a^x f(t)dt = F(x) - F(a)$$

This **defines** an anti-derivative F of f, because

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}(F(x) - F(a)) = f(x)$$

hence

$$\frac{d}{dx}F(x) = f(x)$$

So: if you don't know the anti-derivative of a function f, but have need of it, then this gives you a way of defining one:

$$F(x) = \int_a^x f(t)dt + F(a)$$

3 The Most Important Definition in Calculus

also comes in two forms (one a constant, the other a function):

$$f'(b) = \lim_{h\to 0}\frac{f(b+h) - f(b)}{h}$$

(the derivative, or instantaneous rate of change, at a point $x = b$) or, more generally,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(the derivative, or instantaneous rate of change, at a point x, where x varies over an interval).

We started the course with limits, and this is the most important limit of all. It's an indeterminate limit in the really important cases (that is, when f is continuous): that's because if f is continuous at a point x, then

$$\lim_{h \to 0} f(x+h) = f(x)$$

(that is, f must be continuous at x). Thus the numerator and the denominator in both important definitions are simultaneously going to zero. The question is, what does the indeterminate form yield?

We did a lot of practice with these, and we can actually use a few tricks for computing these. E.g., if we want the derivative function of $f(x) = x^2 - 2x + 3$, we just apply the rule:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) + 3 - (x^2 - 2x + 3)}{h}$$

Expanding,

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h + 3 - (x^2 - 2x + 3)}{h}$$

and then simplifying, we obtain

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \to 0} (2x + h - 2) = 2x - 2$$

by the limit law that says that we can substitute the limit for continuous functions of h (our variable in this problem, and a function which is polynomial). Pretty straightforward in this case.

4 Linearization

Now, once we have the derivative at the point $x = b$,

$$f'(b) = \lim_{h \to 0} \frac{f(b+h) - f(b)}{h}$$

we know the slope of the tangent line – that's the graphical upshot of all of this. Once we know a point $(b, f(b))$ and a slope $f'(b)$ we have a line. That line, the tangent line to the graph of f at the point $x = b$, is known as the linearization of the function f about $x = b$, and sometimes represented

$$L(x) = f(b) + f'(b)(x - b)$$

This serves at least two useful functions:

a. It allows us to approximate f at values of x near to $x = b$: $f(b + h) \approx f(b) + f'(b)h$.

b. It's a tool we use to search for roots of the function f via Newton's method.