Mathematics of Motion II
Projectiles

Table of variables

t	time		
v	velocity,	v_0	initial velocity
a	acceleration		
D	distance		
x	position coordinate,	x_0	initial position
x	horizontal coordinate		
y	vertical coordinate		

Summary of equations

▶ For constant velocity, v_0

$$\begin{aligned} v &= v_0 \\ x &= x_0 + v_0 t \\ D &= v_0 t \end{aligned}$$

▶ For constant acceleration, a

$$\begin{aligned} v &= v_0 + at \\ x &= x_0 + v_0 t + \tfrac{1}{2}at^2 \\ D &= v_0 t + \tfrac{1}{2}at^2 \end{aligned}$$

Projectile motion

Consider motion in Earth's gravity. Examples are baseball and other sports, military weapons like catapults or mortars, and satellites and rockets. In the simplest cases there are *two* coordinates,

$$\begin{aligned} x &= \text{horizontal coordinate} \\ y &= \text{vertical coordinate} \end{aligned}$$

We'll neglect air resistance and aerodynamic forces of drag and lift.

▶ There is no horizontal force so the x component of velocity is constant

$$\begin{aligned} v_x(t) &= v_{0x} \\ x(t) &= x_0 + v_{0x} t \end{aligned}$$

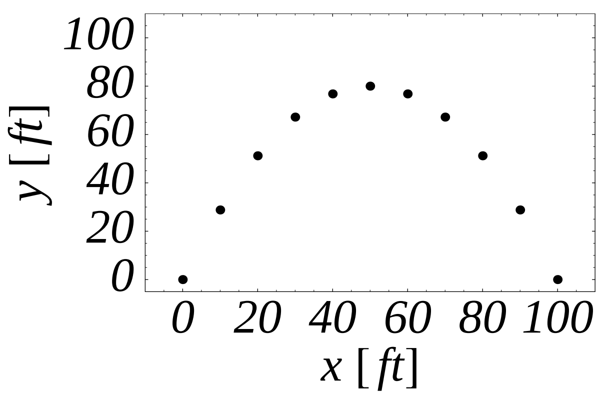
▶ There is a constant downward force, due to gravity. This produces a constant *negative* acceleration for the y coordinate

$$a_y = -g$$

where $g = 9.8\,\text{m/s}^2 = 32\,\text{ft/s}^2$. Therefore

$$\begin{aligned} v_y(t) &= v_{0y} - gt \\ y(t) &= y_0 + v_{0y} t - \tfrac{1}{2}gt^2 \end{aligned}$$

Picture of the trajectory showing the position of the projectile at equal time intervals.



§ The trajectory is a parabola with negative curvature.

§ We can plot a trajectory curve by making a *parametric plot*; that is, plot $y(t)$ versus $x(t)$ using t as an independent parameter. Your graphing calculator should have a parametric plot mode.

Exercises

Exercise 1

A ball is thrown horizontally at 50 mi/hr from a height of 5 ft. Where will it hit the ground?

Exercise 2

The castle is 300 m distant from the catapult. If the initial direction of the velocity vector of the stone is 45 degrees above the horizontal, what initial speed v_0 is required to hit the castle?

(Hint: The initial velocity vector is

$$\mathbf{v}_0 = \widehat{\mathbf{i}} v_0 \cos 45 + \widehat{\mathbf{j}} v_0 \sin 45;$$

that is, $v_{0x} = v_0/\sqrt{2}$ and $v_{0y} = v_0/\sqrt{2}$.)

Exercise 3 – Parametric plots in Mathematica

A parametric plot is a kind of graph—a curve of y versus x—where x and y are known as functions of an independent variable t called *the parameter*. To plot the curve specified by

$$x = f(t) \quad \text{and} \quad y = g(t),$$

the Mathematica command is

```
ParametricPlot[{f[t],g[t]},{t,t1,t2},
  PlotRange->{{x1,x2},{y1,y2}},
  AspectRatio->r]
```

Here `{t1,t2}` is the domain of t, and `{x1,x2}` and `{y1,y2}` are the ranges of x and y. To make the x and y axes have the same scales, `r` should have the numerical value of `(y2-y1)/(x2-x1)`.

Use Mathematica to make the parametric plots below. In each case state in words what the curve is.

(a) $x(t) = t$, $y(t) = t - t^2$.

(b) $x(t) = t$, $y(t) = 1/t$.

(c) $x(t) = \cos(2\pi t)$, $y(t) = \sin(2\pi t)$.

(d) $x(t) = 2\cos(2\pi t)$, $y(t) = 0.5\sin(2\pi t)$.

(e) $x(t) = \cos(2\pi t/3)$, $y(t) = \sin(2\pi t/7)$.

Exercise 4 – Baseball home run

A slugger hits a ball. The speed of the ball as it leaves the bat is $v_0 = 100$ mi/hr $= 147$ ft/sec. Suppose the initial direction is 45 degrees above the horizontal, and the initial height is 3 ft.

(a) Plot y as a function of x, using Mathematica.

```
x[t_]:=   (put the equation for x here)
y[t_]:=   (put the equation for y here)
ParametricPlot[{x[t],y[t]},{t,0,5},
  PlotRange->{{0,350},{0,100}},
  AspectRatio->100/350,
  AxesLabel->{"x (ft)","y (ft)"}]
```

(b) When *precisely* does the ball hit the ground?

(Hint: Use the Mathematica command `FindRoot`. To solve the equation $F(t) = C$, for t, the Mathematica command is

```
FindRoot[F[t]==C,{t,t1}]
```

where t_1 is an initial estimate of the solution.)

(c) Where precisely does the ball hit the ground?

(Hint: Just ask Mathematica for `x[`*the answer to (b)*`]`.)

(d) We have neglected air resistance. Is that a good approximation? Justify your answer.

Exercise 5 – Conservation of energy for a projectile

(a) Consider a projectile, moving under gravity but with negligible air resistance, such as a shot put. Assume these initial values

$$\begin{aligned} x_0 &= 0 \quad &\text{and}\quad v_{0x} &= 10\,\text{m/s} \\ y_0 &= 1.6\,\text{m} \quad &\text{and}\quad v_{0y} &= 8\,\text{m/s} \end{aligned}$$

Use Mathematica or a graphing calculator to make plots of x versus t and y versus t. Hand in sketches of the plots, and remember to show the scales on the axes.

(b) Now plot the total energy (kinetic plus potential) versus t,

$$E(t) = \frac{1}{2}m\left(v_x^2(t) + v_y^2(t)\right) + mgy(t)$$

where $m = 7\,\mathrm{kg}$.

(c) Prove mathematically that E is a constant of the motion.

Exercise 6 – The jumping squirrel

The squirrel wants to jump from A to B. The horizontal distance from A to B is $x = 5\,\mathrm{ft}$, and the vertical distance is $y = 4\,\mathrm{ft}$. If the squirrel jumps with an initial speed of $20\,\mathrm{ft/s}$, at what angle to the horizontal should it jump?

Exercise 7 – The tractrix

The tractrix is the curve followed by an object pulled by a string and sliding on a frictional surface. The instantaneous velocity of the object is always in the direction of the string, and the string length remains constant.

The initial position of the object is $(x_0, y_0) = (0, a)$; that is, the object is initially on the y axis at distance a from the origin. The length of the string is a. The other end of the string is initially at the origin; thereafter it moves with constant velocity v along the x axis. Parametric equations for the tractrix, *i.e.,* the trajectory of the pulled object in the xy plane, are

$$\begin{aligned} x &= vt - a\tanh\left(\frac{vt}{a}\right), \\ y &= \frac{a}{\cosh\left(\frac{vt}{a}\right)}. \end{aligned}$$

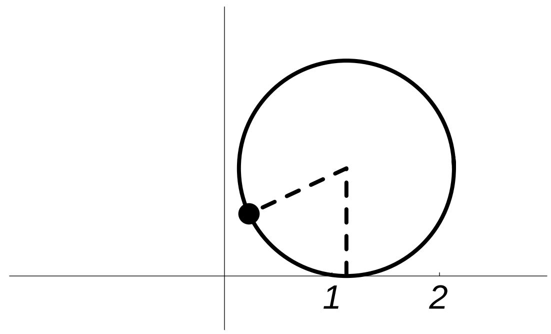
(a) Plot the tractrix—the curve in the xy plane—for unit values of a and v.

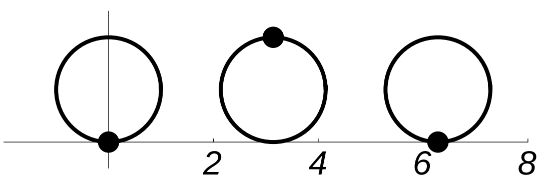
(b) When the end of the string is at $(2, 0)$ (with $a = 1$) where is the dragged object?

Exercise 8 – The cycloid

Consider a circle of radius R that rolls without slipping on a line. For example, the circle could be a bicycle tire, and the straight line a road. We'll determine the motion of a point P fixed on the circle. For example, if there is a white dot painted on the side of the bicycle tire, how does that white dot move as the tire rolls along the road?

Figure 1 shows the circle at three different times. At $t = 0$ the point P is at the origin. The circle rolls on the x axis. At $t = 1$ the point P is at the top of the circle, and at time $t = 2$ it is back at the bottom again.





Now consider an arbitrary time t, when the line connecting P to the center of the circle is at angle θ to the vertical, as shown in Figure 2. At this time t the center of the circle is at $x = R\theta$: because $R\theta$ is the arclength, and the arc has the same length as the distance traveled along the road if the circle rolls without slipping. (Note: the angle θ is measured in radians.)

(a) Use trigonometry to determine the x coordinate of the point P in Figure 2. (The result depends on $\sin\theta$.)

(b) Use trigonometry to determine the y coordinate of the point P in Figure 2. (The result depends on $\cos\theta$.)

(c) Make a plot of the curve traced out by the point P in the xy space. (Set $R = 1$.)

(Hint: From (a) and (b) you have equations for x and y as functions of the parameter θ,

$$x = f(\theta) \quad \text{and} \quad y = g(\theta),$$

so use the *parametric plot* mode of your graphing calculator to plot y versus x.)

The cycloid. The curve traced out by P as the circle rolls along the line is called a cycloid. If you're stuck at this point, try looking up "cycloid curve" on the Internet.

(d) From the result of (c) prove that when P is in contact with the line that the circle rolls on (*i.e.,* the x axis in Figure 1 or 2) the instantaneous velocity of P is 0. In other words, when you ride a bicycle, the point on either tire in contact with the road is instantaneously at rest. So how can you be moving!?

Answers Nov 1, 2001

Exercise 1

Where will the ball hit the ground?

Exercise 2

What initial speed is required to hit the castle?

Exercise 3 – Parametric plots

(a) Sketch the graph.
What is the curve?

(b) Sketch the graph.
What is the curve?

(c) Sketch the graph.
What is the curve?

(d) Sketch the graph.
What is the curve?

(e) Sketch the graph.
What is the curve?

Exercise 4 – Baseball home run

(a) Sketch the trajectory; include scales on the axes.

(b) Time when the ball hits the ground =

(c) Distance where the ball hits the ground =

(d) Is it a good approximation? Explain.

Exercise 5 – Conservation of energy

(a) Sketch plots of $x(t)$ and $y(t)$; include scales on the axes.

(b) Sketch a plot of E versus t.

(c) Proof:

Exercise 6 – The jumping squirrel

At what angle should the squirrel jump to reach the other branch?

Exercise 7 – The tractrix

(a) Sketch a plot of the tractrix.

(b) When the end of the string is at $(0, 2)$ where is the dragged object?