Mathematics of Motion I
One Dimensional Motion

Table of variables

t	time		
v	velocity,	v_0	initial velocity
a	acceleration		
D	distance		
x	position coordinate,	x_0	initial position
x	horizontal coordinate		
y	vertical coordinate		

Summary of equations for one dimensional motion

• For constant velocity, v_0

$$\begin{aligned} v &= v_0 \\ x &= x_0 + v_0 t \\ D &= v_0 t \end{aligned}$$

• For constant acceleration, a

$$\begin{aligned} v &= v_0 + at \\ x &= x_0 + v_0 t + \frac{1}{2}at^2 \\ D &= v_0 t + \frac{1}{2}at^2 \end{aligned}$$

• For a falling object (neglecting air resistance) let y be the height above ground level. Then

$$\begin{aligned} v &= v_0 - gt \\ y &= y_0 + v_0 t - \frac{1}{2}gt^2 \end{aligned}$$

The vertical acceleration is $a = -g$ (negative meaning downward) where $g = 9.8\,\mathrm{m/s^2}$ or $g = 32\,\mathrm{ft/s^2}$.

• For *arbitrary* motion, let $x(t) =$ position (coordinate) as a function of time t.

$$\begin{aligned} v(t) &= \frac{dx}{dt} \quad \text{the derivative of } x \text{ wrt } t \\ a(t) &= \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad \text{the derivative of } v \text{ wrt } t \end{aligned}$$

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Exercises

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1 – ***Race car.*** A race car accelerates from 0 to 60 mph in 6 seconds.

(a) What is the acceleration? Express a in ft/sec^2.

(b) How far does the car travel in that 6 seconds?

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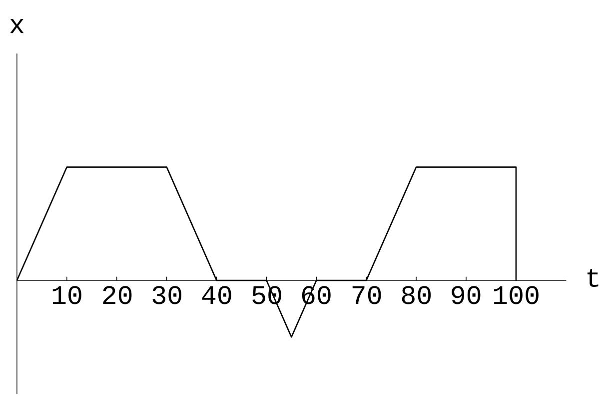
2 – The graph below describes a short ride in a car on a straight road, showing velocity v versus time t.



Explain in words what is happening at thse times:

(a) from 0 to 10 s
(b) from 10 to 30 s
(c) from 30 to 40 s
(d) from 40 to 50 s
(e) from 50 to 60 s
(f) at $t = 100$ s.

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3 – ***Platform diving.*** A diver jumps off a 10 m platform. How many seconds does she have to do all her twists and flips before she enters the water? (Assume her initial upward velocity is 0.)

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4 – ***Conservation of energy.*** A rock falls from a cliff 100 m high, and air resistance can be neglected.

(a) Plot y (=height) versus t (=time). Use Mathematica and hand in a sketch of the result.

(b) Plot v (=speed) versus t.

(c) Plot $v^2/2 + gy$ versus t. Describe the result in words.

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5 – ***Braking.*** (a) You are driving on the highway at 60 mph (=88 ft/s). There is an accident ahead, so you brake hard, decelerating at $0.3\,g = 9.6\,\mathrm{ft/s^2}$.

(a) How much time does it take to stop?

(b) How far will you travel before stopping?

(c) How far would you travel if your initial speed were 75 mph, assuming the same deceleration?

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6 – A ball rolling down an inclined plane has constant acceleration a. It is released from rest. u is a unit of length.

During the first second the ball travels a distance $1\,u$ on the inclined plane.

(a) How far does it travel during the second second?

(b) How far does it travel during the third second?

(c) How far does it travel during the tenth second?

(d) How far did it travel altogether after 10 seconds?

(e) What is a?

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7 – ***Mathematica reminder.*** Define the function $f(x) = 2x^2 - 6x + 4$. Make a graph of $f(x)$ versus x in the range from $x = 0$ to $x = 4$. Hand in a printout of the graph.

To define the function:

```
f[x_]:=2x^2-6x+4
```

To make the plot:

```
Plot[f[x],{x,0,4},
  PlotRange->{{0,5},{-1,15}}]
```

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8 – ***Derivatives with Mathematica.*** Continuing with the same function $f(x)$ from the previous exercise, determine the derivative function $f'(x)$ and make a graph of $f'(x)$ versus x. Hand in a printout of the graph. Prove that where the slope

of $f(x)$ is positive, $f'(x) > 0$; and where the slope of $f(x)$ is negative, $f'(x) < 0$.

To define the derivative function in Mathematica:

```
fp[x_]:=f'[x]
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9 – Consider the function

$$g(t) = te^{-t}.$$

(In Mathematica, `g[t_]:=t*Exp[-t]`.)

(a) Plot $g(t)$ versus t for $t \geq 0$ and hand in a printout of the graph.

(b) Plot $g'(t)$ versus t for $t \geq 0$ and hand in a printout of the graph.

(c) Recall that $g'(t)$ is the slope in a graph of $g(t)$. Prove that $g'(t)$ is positive where the slope of g is positive and $g'(t)$ is negative where the slope of g is negative.

(d) For what t is $g'(t) = 0$? Where is that point on the graph of $g(t)$?

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10 – A car leaving a stop sign at $t = 0$ has velocity

$$v(t) = \frac{Ct^2}{t^2 + 4}$$

where t is in seconds and $C = 30\,\mathrm{mi/hr}$ (=44 ft/s).

(a) Use Mathematica to plot v versus t. Hand in a sketch of the graph.

(b) At what time is the velocity 25 mi/hr?

(c) What is the maximum acceleration? (Use Mathematica to plot $a(t) = v'(t)$.) Express the result in $\mathrm{ft/s^2}$.

(d) At what time does maximum acceleration occur?

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11 – ***Air resistance and terminal speed.*** Galileo made the brilliant deduction that all objects would fall with the same acceleration, except for air resistance. How much actual difference is there between falling objects, taking into account air resistance?

According to Newton's theory of air resistance (drag force $\propto v^2$) the height of an object dropped from rest is, at time t,

$$y(t) = y_0 - \frac{v_T^2}{g} \ln \left[\cosh \left(\frac{gt}{v_T} \right) \right]$$

where y_0 is the initial height and v_T is the terminal speed. (The speed increases until the object is falling at the terminal speed, and thereafter remains constant at v_T. More precisely, v approaches v_T *asymptotically.*)

(a) Use Mathematica or some other computer graphing program, or a graphing calculator, to plot $y(t)$ for a baseball ($v_T = 42\,\mathrm{m/s}$) and a ping-pong ball ($v_T = 9\,\mathrm{m/s}$). Let the initial height be $y_0 = 10\,\mathrm{m}$.

(b) Determine the difference of times when the 2 balls hit the ground (at $y = 0$).

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12 – ***Terminal speed of a baseball***

Use the equation in the previous problem.

(a) At what time is the speed of the baseball 90% of v_T? (Here assume y_0 is large enough that the ball does not hit the ground.) For a baseball, $v_T = 42\,\mathrm{m/s}$.

(b) How far has the baseball fallen when its speed reaches 90% of v_T?

Hint: The velocity is $v(t) = y'(t)$. There are several ways to calculate the derivative: analytically by calculus or by Mathematica, or numerically by Excel.

Answer Sheet

Hand in as many as possible today, on the green sheet, and do the rest for homework on the pink sheet.

1. (a) acceleration =

   (b) distance =

2. (a) For $t \in (0, 10)$:

   (b) For $t \in (10, 30)$:

   (c) For $t \in (30, 40)$:

   (d) For $t \in (40, 50)$:

   (e) For $t \in (50, 60)$:

   (f) At $t = 100\,\mathrm{s}$:

3. Time of the dive =

4. (a) Graph of $y(t)$

   (b) Graph of $v(t)$

   (c) Graph of $v^2/2 + gy(t)$. Describe the graph in words.

5. (a) Time required to stop =

   (b) distance while stopping =

   (c) distance if initial speed is $75\,\mathrm{mph}$

6. (a) Distance during the 2^{nd} second=

   (b) Distance during the 3^{rd} second=

   (c) Distance during the 10^{th} second=

   (d) Total distance=

   (e) $a =$

7. Graph of $f(x)$

8. Graph of $f'(x)$

9. (a) Graph of $g(t)$

   (b) Graph of $g'(t)$

   (c) Proof:

   (d) $g' = 0$ at $t =$

   What point is that on the graph in (a)?

10. (a) Graph of $v(t)$

    (b) The time when $v = 25\,\mathrm{mph}$ is

    (c) Maximum acceleration =

    (d) Time of maximum acceleration =

11. Air resistance.

    (a) Graph of $y(t)$

    (b) Time difference=