Calculus I: General Concepts

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1 Functions

- The function zoo: polynomial, rational, trig, power, root, etc.
- domain, range
- symmetry (even, odd)
- asymptotes (vertical, horizontal, slant)
- increasing, decreasing
- concavity - up like a bowl, down like an umbrella
- continuity and discontinuities (connectedness versus holes or jumps)
- differentiability (smoothness)
- one-to-one, invertible (pass horizontal and vertical line tests)
- compositions of functions
- transformations (shifts and stretches) of functions

2 Limits

- limit laws
- tangent versus secant lines
- limits at infinity, and infinite limits
- continuity and discontinuity
- Intermediate value theorem
- pinching or squeeze theorem

3 Derivatives and differentiation

- The limit definition of the derivative (the most important definition in calculus!):
  - at a point:
  $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
  - function:
  $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  which represents the slope of the tangent line to the graph at either $x = a$ or x in general.
- relationship between the size and sign of the derivative of f and the shape of the graph of f
- sum, difference, product, quotient rules
- chain rule (derivatives of compositions)
- implicit differentiation
- higher derivatives
- linearization
- related rates problems
- optimization problems
- anti-derivatives

4 Optimization

- Setting up the problem is the hard part! Draw pictures....
- extrema (maxima, minima; local and global)
- first and second derivative tests
- check endpoints!

5 Integrals and Integration

- Motivation (the beginning, but not the end!): areas
- Riemann sums: using rectangles via approximation rules to estimate areas (left, right, mid-point, trapezoidal – ultimately Simpson's rule).
- **A rule to remember:** When you have two estimates for a quantity, you have a third – some average of the two. The average may be weighted, depending on the confidence you have in the two estimates (e.g. Simpson's rule, which gives twice the confidence to the midpoint over the trapezoidal rule).
- Taking a limit of a Riemann sum to define the area exactly.
- **Fundamental Theorem of Calculus** (tying derivates together with integrals): Suppose f is continuous on $[a, b]$.

  a. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

  b.
  $$\int_a^b f(x)dx = F(b) - F(a)$$
  where F is any anti-derivative of f.

- substitution rule (the chain rule in reverse)
- integrals as more general than simply areas: so
  $$S = \int dS$$
  means that in computing quantity S (area, length, volume, time, dollars, work, mass, etc.) we simply add up lots of tiny amounts (infinitesimals) of S. Examples:

  – areas between two curves

  – An example of an important extension of integrals is volume calculations:
  $$V = \int dV = \int A(x)dx = \int C(x)dxh(x)$$
  We might slice into cross-sections of known area $A(x)$, or we might chop into cylinders of known circumference $C(x)$ and height $h(x)$. The point is to model dV appropriately, whatever the case.

  – Here's the sort of application one might do someday as an engineer. The time to arrive at State Route 126 from Exit 3 on I-75 is
  $$T = \int dT = \int_{\text{Exit 3 on I-75}}^{\text{Exit10A:Route 126}} \frac{dx}{v(x)}$$
  where $v(x)$ is the average automobile velocity at every point along the road from point a to point b. (dt was calculated using the DiRT formula - don't forget the dirt formula!). To compute a time, we add up lots of little times dT, which are computed from an instantaneous velocity over a very short distance dx.

- Use symmetry whenever possible, e.g. to help calculate integrals
- definite versus indefinite integrals: numbers versus families of functions.