6 APPLICATIONS OF THE INTEGRAL

The CAT scan is based on tomography, a mathematical technique for combining a large series of X-rays of the body taken at different angles into a single cross-sectional image.

The integral, like the derivative, has a wide variety of applications. In the previous chapter, we used the integral to compute areas under curves and net change. In this chapter, we discuss some of the other quantities that are represented by integrals, including volume, average value, work, total mass, population, and fluid flow.

6.1 Area Between Two Curves

One goal of integration is to compute the area of regions enclosed by curves. The definite integral does not fully accomplish this goal because $\int_a^b f(x)\,dx$ only gives us the signed area of the region between the graph of a function $f(x)$ and the x-axis. Suppose, however, that the region can be expressed as the region between the graphs of two functions $f(x)$ and $g(x)$ over $[a,b]$ where $f(x) \geq g(x)$. Then the actual area of the region (not the signed area) is equal to the integral of $f(x) - g(x)$:

$$\text{Area between the graphs} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx \qquad \boxed{1}$$

We can justify this formula in the case that $f(x) \geq g(x) \geq 0$ by referring to Figure 1. we see that the region between the graphs is obtained by removing the region under $y = g(x)$ from the region under $y = f(x)$.

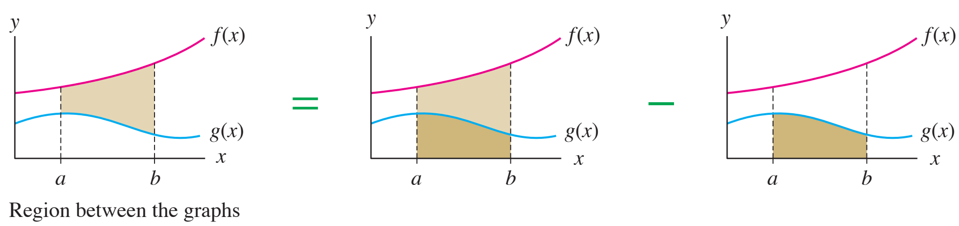


FIGURE 1 The area between the graphs is a difference of two areas.

■ **EXAMPLE 1** Calculate the area of the region between the graphs of

$$f(x) = x^2 - 4x + 10 \qquad \text{and} \qquad g(x) = 4x - x^2$$

over $[1,3]$.

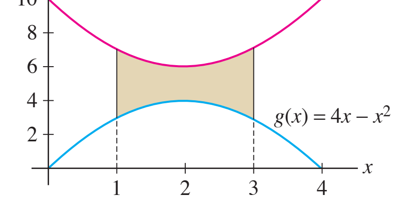


FIGURE 2 Region between the graphs of $f(x) = x^2 - 4x + 10$ and $g(x) = 4x - x^2$ over $[1, 3]$.

Solution To calculate the area between the graphs, we must first determine which graph lies on top. Figure 2 shows that $f(x) \geq g(x)$. We can verify this without appealing to the graph by completing the square:

$$f(x) - g(x) = (x^2 - 4x + 10) - (4x - x^2) = 2x^2 - 8x + 10 = 2(x-2)^2 + 2 > 0$$

By Eq. (1), the area between the graphs is

$$\int_1^3 \big(f(x) - g(x)\big)\,dx = \int_1^3 \big((x^2 - 4x + 10) - (4x - x^2)\big)\,dx$$

$$= \int_1^3 (2x^2 - 8x + 10)\,dx = \left(\frac{2}{3}x^3 - 4x^2 + 10x\right)\Bigg|_1^3 = 12 - \frac{20}{3} = \frac{16}{3} \quad ■$$

Before continuing with more examples, let us use Riemann sums to explain why Eq. (1) remains valid if $f(x) \geq g(x)$ but $f(x)$ and $g(x)$ are not assumed to be positive:

$$\int_a^b \big(f(x) - g(x)\big)\,dx = \lim_{\|P\|\to 0} R(f - g, P, C) = \lim_{N\to\infty} \sum_{i=1}^{N} \big(f(c_i) - g(c_i)\big)\Delta x_i$$

Recall that P denotes a partition of $[a, b]$:

$$\text{Partition } P\text{:} \quad a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

$C = \{c_1, \ldots, c_N\}$ is a choice of sample points where $c_i \in [x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$. The ith term in the Riemann sum is equal to the area of a thin vertical rectangle of height $(f(c_i) - g(c_i))$ and width Δx_i (Figure 3):

$$\big(f(c_i) - g(c_i)\big)\Delta x_i = \text{height} \times \text{width}$$

Therefore, $R(f - g, P, C)$ is an approximation to the area between the graphs using thin vertical rectangles. As the norm $\|P\|$ (the maximum width of the rectangles) approaches zero, the Riemann sum converges to the area between the graphs and we obtain Eq. (1).

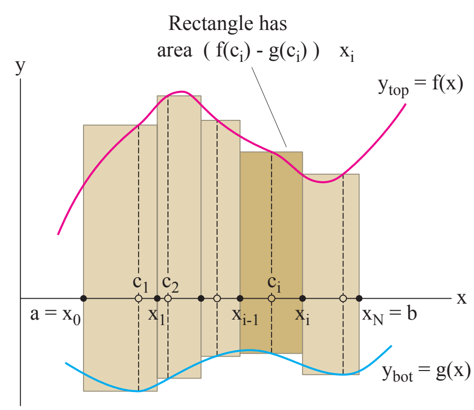


FIGURE 3 The ith rectangle has width $\Delta x_i = x_i - x_{i-1}$ and height $f(c_i) - g(c_i)$.

To help identify the functions, we sometimes denote the upper graph by $y_{\text{top}} = f(x)$ and the lower graph by $y_{\text{bot}} = g(x)$:

Keep in mind that $(y_{\text{top}} - y_{\text{bot}})$ *is the height of a thin vertical slice of the region.*

$$\text{Area between the graphs} = \int_a^b (y_{\text{top}} - y_{\text{bot}})\,dx = \int_a^b \big(f(x) - g(x)\big)\,dx \qquad \boxed{2}$$

EXAMPLE 2 Find the area between the graphs of $f(x) = x^2 - 5x - 7$ and $g(x) = x - 12$ over $[-2, 5]$.

Solution To calculate the area, we must first determine which graph lies on top.

***Step 1.* Sketch the region (especially, find any points of intersection).**
We know that $y = f(x)$ is a parabola with y-intercept -7 and $y = g(x)$ is a line with y-intercept -12. To determine where the graphs intersect, we solve $f(x) = g(x)$:

$$x^2 - 5x - 7 = x - 12 \qquad \text{or} \qquad x^2 - 6x + 5 = (x-1)(x-5) = 0$$

Thus, the points of intersection are $x = 1, 5$ (Figure 4).

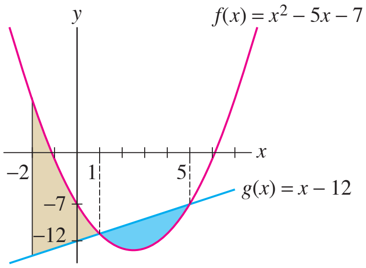


FIGURE 4

***Step 2.* Set up the integrals and evaluate.**
Figure 4 shows that

$$f(x) \ge g(x) \text{ on } [-2, 1] \qquad \text{and} \qquad g(x) \ge f(x) \text{ on } [1, 5]$$

Therefore, we write the area as a sum of integrals over the two intervals:

$$\begin{aligned}
\int_{-2}^{5} (y_{\text{top}} - y_{\text{bot}})\,dx &= \int_{-2}^{1} \big(f(x) - g(x)\big)\,dx + \int_{1}^{5} \big(g(x) - f(x)\big)\,dx \\
&= \int_{-2}^{1} \big((x^2 - 5x - 7) - (x - 12)\big)\,dx + \int_{1}^{5} \big((x - 12) - (x^2 - 5x - 7)\big)\,dx \\
&= \int_{-2}^{1} (x^2 - 6x + 5)\,dx + \int_{1}^{5} (-x^2 + 6x - 5)\,dx \\
&= \left(\frac{1}{3}x^3 - 3x^2 + 5x\right)\Bigg|_{-2}^{1} + \left(-\frac{1}{3}x^3 + 3x^2 - 5x\right)\Bigg|_{1}^{5} \\
&= \left(\frac{7}{3} - \frac{(-74)}{3}\right) + \left(\frac{25}{3} - \frac{(-7)}{3}\right) = \frac{113}{3}
\end{aligned}$$

In Example 2, we found the intersection points of $y = f(x)$ and $y = g(x)$ algebraically. For more complicated functions, it may be necessary to use a computer algebra system.

EXAMPLE 3 Calculating Area by Dividing the Region Find the area of the region bounded by the graphs of $y = \dfrac{8}{x^2}$, $y = 8x$, and $y = x$.

Solution

***Step 1.* Sketch the region (especially, find any points of intersection).**
The curve $y = \dfrac{8}{x^2}$ cuts off a region in the sector between the two lines $y = 8x$

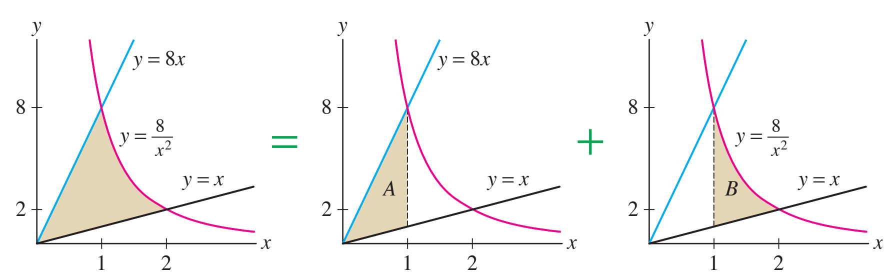


FIGURE 5 Area bounded by $y = \dfrac{8}{x^2}$, $y = 8x$, and $y = x$ as a sum of two areas.

and $y = x$ (Figure 5). To find the intersection of $y = \frac{8}{x^2}$ and $y = 8x$, we solve

$$\frac{8}{x^2} = 8x \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad x = 1$$

To find the intersection of $y = \frac{8}{x^2}$ and $y = x$, we solve

$$\frac{8}{x^2} = x \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2$$

***Step 2.* Set up the integrals and evaluate.**
Figure 5 shows that $y_{\text{bot}} = x$, but y_{top} changes at $x = 1$ from $y_{\text{top}} = 8x$ to $y_{\text{top}} = \frac{8}{x^2}$. Therefore, we break up the regions into two parts, A and B, and compute their areas separately:

$$\text{Area of } A = \int_0^1 (y_{\text{top}} - y_{\text{bot}})\, dx = \int_0^1 (8x - x)\, dx = \int_0^1 7x\, dx = \frac{7}{2}x^2\Big|_0^1 = \frac{7}{2}$$

$$\text{Area of } B = \int_1^2 (y_{\text{top}} - y_{\text{bot}})\, dx = \int_1^2 \left(\frac{8}{x^2} - x\right) dx = \left(-\frac{8}{x} - \frac{1}{2}x^2\right)\Big|_1^2 = \frac{5}{2}$$

The total area bounded by the curves is the sum $\frac{7}{2} + \frac{5}{2} = 6$. ■

Integration Along the y-Axis

Suppose we are given x as a function of y, say, $x = g(y)$. What is the meaning of the integral $\int_c^d g(y)\, dy$? This integral may be interpreted as *signed area*, where regions to the *right* of the y-axis have positive area and regions to the *left* have negative area:

$$\int_c^d g(y)\, dy = \text{signed area between graph and } y\text{-axis for } c \le y \le d$$

Figure 6(A) shows the graph of $g(y) = y^2 - 1$. The region to the left of the y-axis has negative signed area. The integral is equal to the signed area:

$$\underbrace{\int_{-2}^2 (y^2 - 1)\, dy}_{\substack{\text{Area to the right of } y\text{-axis minus} \\ \text{area to the left of } y\text{-axis}}} = \left(\frac{1}{3}y^3 - y\right)\Big|_{-2}^2 = \frac{4}{3}$$

More generally, if $g_2(y) \ge g_1(y)$ as in Figure 6(B), then the graph of $x = g_2(y)$ lies to the right of the graph of $x = g_1(y)$. As a reminder, we write $x_{\text{right}} = g_2(y)$ and $x_{\text{left}} = g_1(y)$. The area between the two graphs for $c \le y \le d$ is equal to

$$\text{Area between the graphs} = \int_c^d \big(g_2(y) - g_1(y)\big)\, dy = \int_c^d (x_{\text{right}} - x_{\text{left}})\, dy \qquad \boxed{3}$$

In this case, the Riemann sums approximate the area by thin horizontal rectangles of width $x_{\text{right}} - x_{\text{left}}$ and height Δy.

■ **EXAMPLE 4** Calculate the area between the graphs of $g_1(y) = y^3 - 4y$ and $g_2(y) = y^3 + y^2 + 8$ for $-2 \le y \le 2$.

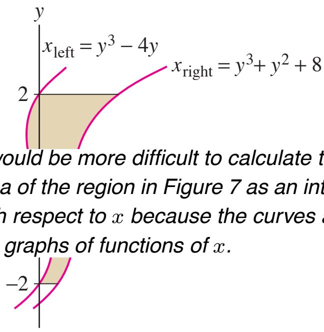
Solution We confirm that $g_2(y) \ge g_1(y)$ as shown in Figure 7:

$$g_2(y) - g_1(y) = (y^3 + y^2 + 8) - (y^3 - 4y) = y^2 + 4y + 8 = (y+2)^2 + 4 > 0$$

Therefore, $x_{\text{right}} = g_2(y)$ and $x_{\text{left}} = g_1(y)$, and the area is

$$\int_{-2}^{2} (x_{\text{right}} - x_{\text{left}})\, dy = \int_{-2}^{2} (y^2 + 4y + 8)\, dy = \left(\frac{1}{3}y^3 + 2y^2 + 8y\right)\Bigg|_{-2}^{2}$$

$$= \frac{80}{3} - \frac{-32}{3} = \frac{112}{3}$$ ■



It would be more difficult to calculate the area of the region in Figure 7 as an integral with respect to x because the curves are not graphs of functions of x.

FIGURE 7 Region between $g_1(x) = y^3 - 4y$ and $g_2(x) = y^3 + y^2 + 8$ for $-2 \le y \le 2$.

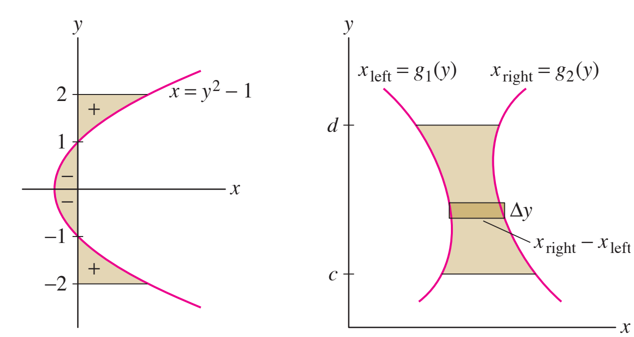
6.1 SUMMARY

- If $f(x) \ge g(x)$ on $[a, b]$, then the area between the graphs of f and g over $[a, b]$ is

$$\text{Area between graphs} = \int_a^b \big(f(x) - g(x)\big)\, dx = \int_a^b \big(y_{\text{top}} - y_{\text{bot}}\big)\, dx$$

- To calculate the area between two graphs $y = f(x)$ and $y = g(x)$, sketch the region to find y_{top}. If necessary, find points of intersection by solving $f(x) = g(x)$.
- The integral along the y-axis, $\int_c^d g(y)\, dy$, is equal to the signed area between the graph and the y-axis for $c \le y \le d$, where area to the right of the y-axis is positive and area to the left is negative.
- If $g_2(y) \ge g_1(y)$, then the graph of $x = g_2(y)$ lies to the right of the graph of $x = g_1(y)$ and the area between the graphs for $c \le y \le d$ is

$$\text{Area between graphs} = \int_c^d \big(g_2(y) - g_1(y)\big)\, dy = \int_c^d \big(x_{\text{right}} - x_{\text{left}}\big)\, dy$$



(A) Region between $x = y^2 - 1$ and the y-axis

(B) Region between $x = g_2(y)$ and $x = g_1(y)$

FIGURE 6

6.1 EXERCISES

Preliminary Questions

1. What is the area interpretation of $\int_a^b (f(x) - g(x))\, dx$ if $f(x) \geq g(x)$?

2. Is $\int_a^b (f(x) - g(x))\, dx$ still equal to the area between the graphs of f and g if $f(x) \geq 0$ but $g(x) \leq 0$?

3. Suppose that $f(x) \geq g(x)$ on $[0, 3]$ and $g(x) \geq f(x)$ on $[3, 5]$. Express the area between the graphs over $[0, 5]$ as a sum of integrals.

4. Suppose that the graph of $x = f(y)$ lies to the left of the y-axis. Is $\int_a^b f(y)\, dy$ positive or negative?

Exercises

1. Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$ (Figure 8).

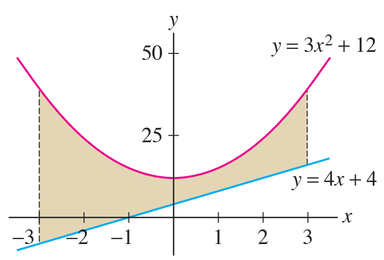


FIGURE 8

2. Compute the area of the region in Figure 9(A), which lies between $y = 2 - x^2$ and $y = -2$ over $[-2, 2]$.

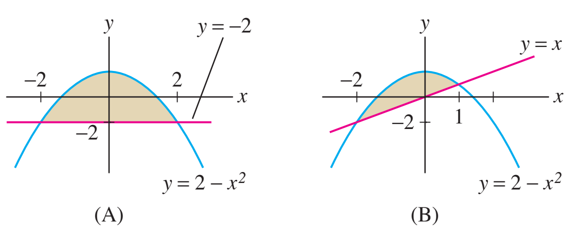


FIGURE 9

3. Let $f(x) = x$ and $g(x) = 2 - x^2$ [Figure 9(B)].

(a) Find the points of intersection of the graphs.

(b) Find the area enclosed by the graphs of f and g.

4. Let $f(x) = 8x - 10$ and $g(x) = x^2 - 4x + 10$.

(a) Find the points of intersection of the graphs.

(b) Compute the area of the region *below* the graph of f and *above* the graph of g.

In Exercises 5–7, find the area between $y = \sin x$ and $y = \cos x$ over the interval. Sketch the curves if necessary.

5. $\left[0, \frac{\pi}{4}\right]$ **6.** $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ **7.** $[0, \pi]$

In Exercises 8–10, let $f(x) = 20 + x - x^2$ and $g(x) = x^2 - 5x$.

8. Find the area between the graphs of f and g over $[1, 3]$.

9. Find the area of the region enclosed by the two graphs.

10. Compute the area of the region between the two graphs over $[4, 8]$ as a sum of two integrals.

11. Find the area between $y = e^x$ and $y = e^{2x}$ over $[0, 1]$.

12. Find the area of the region bounded by $y = e^x$ and $y = 12 - e^x$ and the y-axis.

13. Sketch the region bounded by $y = \frac{1}{\sqrt{1 - x^2}}$ and $y = -\frac{1}{\sqrt{1 - x^2}}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and find its area.

14. Sketch the region bounded by $y = \sec^2 x$ and $y = 2$ and find its area.

In Exercises 15–18, find the area of the shaded region in the figure.

15.

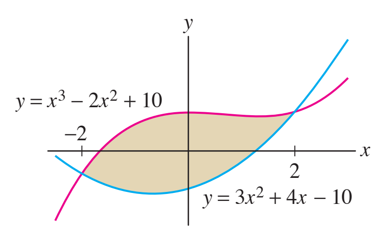


FIGURE 10

16.

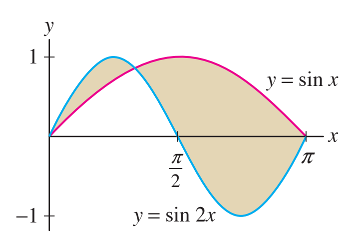


FIGURE 11

17.

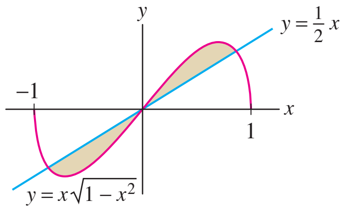


FIGURE 12

18.

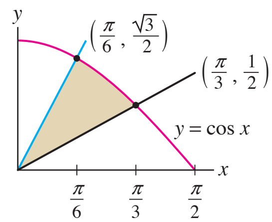


FIGURE 13

19. Find the area of the region enclosed by the curves $y = x^3 - 6x$ and $y = 8 - 3x^2$.

20. Find the area of the region enclosed by the *semicubical parabola* $y^2 = x^3$ and the line $x = 1$.

In Exercises 21–22, find the area between the graphs of $x = \sin y$ *and* $x = 1 - \cos y$ *over the given interval (Figure 14).*

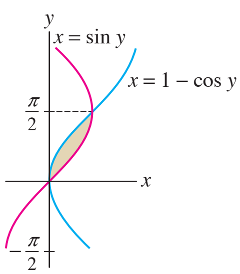


FIGURE 14

21. $0 \le y \le \frac{\pi}{2}$

22. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

23. Find the area of the region lying to the right of $x = y^2 + 4y - 22$ and the left of $x = 3y + 8$.

24. Find the area of the region lying to the right of $x = y^2 - 5$ and the left of $x = 3 - y^2$.

25. Calculate the area enclosed by $x = 9 - y^2$ and $x = 5$ in two ways: as an integral along the y-axis and as an integral along the x-axis.

26. Figure 15 shows the graphs of $x = y^3 - 26y + 10$ and $x = 40 - 6y^2 - y^3$. Match the equations with the curve and compute the area of the shaded region.

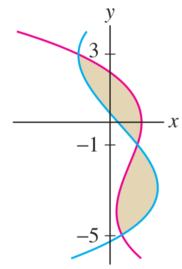


FIGURE 15

In Exercises 27–28, find the area of the region using the method (integration along either the x- or y-axis) that requires you to evaluate just one integral.

27. Region between $y^2 = x + 5$ and $y^2 = 3 - x$

28. Region between $y = x$ and $x + y = 8$ over $[2, 3]$

In Exercises 29–45, sketch the region enclosed by the curves and compute its area as an integral along the x- or y-axis.

29. $y = 4 - x^2, \quad y = x^2 - 4$

30. $y = x^2 - 6, \quad y = 6 - x^3, \quad y$-axis

31. $x = \sin y, \quad x = \frac{2}{\pi}y$

32. $x + y = 4, \quad x - y = 0, \quad y + 3x = 4$

33. $y = 3x^{-3}, \quad y = 4 - x, \quad y = \frac{x}{3}$

34. $y = 2 - \sqrt{x}, \quad y = \sqrt{x}, \quad x = 0$

35. $y = x\sqrt{x-2}, \quad y = -x\sqrt{x-2}, \quad x = 4$

36. $y = |x|, \quad y = x^2 - 6$

37. $x = |y|, \quad x = 6 - y^2$

38. $x = |y|, \quad x = 1 - |y|$

39. $x = 12 - y, \quad x = y, \quad x = 2y$

40. $x = y^3 - 18y, \quad y + 2x = 0$

41. $x = 2y, \quad x + 1 = (y - 1)^2$

42. $x + y = 1, \quad x^{1/2} + y^{1/2} = 1$

43. $y = 6, \quad y = x^{-2} + x^2$ (in the region $x > 0$)

44. $y = \cos x, \quad y = \cos(2x), \quad x = 0, \quad x = \frac{2\pi}{3}$

45. $y = \sin x, \quad y = \csc^2 x, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}$

46. CAS Plot $y = \frac{x}{\sqrt{x^2+1}}$ and $y = (x-1)^2$ on the same set of axes. Use a computer algebra system to find the points of intersection numerically and compute the area between the curves.

47. Sketch a region whose area is represented by

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - |x|\right) dx$$

and evaluate using geometry.

48. NEW Athletes 1 and 2 run in the same direction along a straight track with velocities $v_1(t)$ and $v_2(t)$ (in m/s) as shown in Figure 16.

(a) Which of the following is a correct interpretation of the area between the graphs of $v_1(t)$ and $v_2(t)$ over the time interval $[0, 10]$? Explain.

i. The distance between athletes 1 and 2 at time $t = 10$ s.

ii. The difference in the distance traveled by the athletes 1 and 2 over the time interval $[0, 10]$.

(b) Does Figure 16 give us enough information to determine who is ahead at time $t = 10$ s?

(c) If the athletes begin at the same time and place, who is ahead at $t = 25$ s?

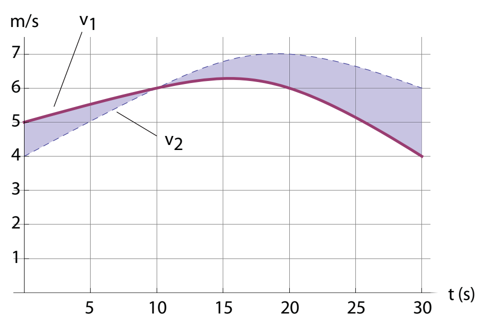


FIGURE 16

49. Express the area (not signed) of the shaded region in Figure 17 as a sum of three integrals involving the functions f and g.

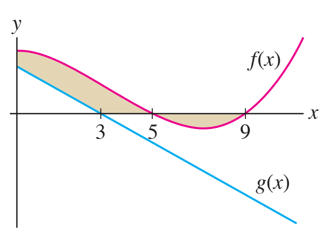


FIGURE 17

50. Find the area enclosed by the curves $y = c - x^2$ and $y = x^2 - c$ as a function of c. Find the value of c for which this area is equal to 1.

51. Set up (but do not evaluate) an integral that expresses the area between the circles $x^2 + y^2 = 2$ and $x^2 + (y-1)^2 = 1$.

52. Set up (but do not evaluate) an integral that expresses the area between the graphs of $y = (1+x^2)^{-1}$ and $y = x^2$.

53. CAS Find a numerical approximation to the area above $y = 1 - (x/\pi)$ and below $y = \sin x$ (find the points of intersection numerically).

54. CAS Find a numerical approximation to the area above $y = |x|$ and below $y = \cos x$.

55. CAS Use a computer algebra system to find a numerical approximation to the number c (besides zero) in $[0, \frac{\pi}{2}]$, where the curves $y = \sin x$ and $y = \tan^2 x$ intersect. Then find the area enclosed by the graphs over $[0, c]$.

56. The back of Jon's guitar (Figure 18) has a length 19 in. He measured the widths at 1-in. intervals, beginning and ending $\frac{1}{2}$ in. from the ends, obtaining the results

$$6, 9, 10.25, 10.75, 10.75, 10.25, 9.75, 9.5, 10, 11.25,$$
$$12.75, 13.75, 14.25, 14.5, 14.5, 14, 13.25, 11.25, 9$$

Use the midpoint rule to estimate the area of the back.

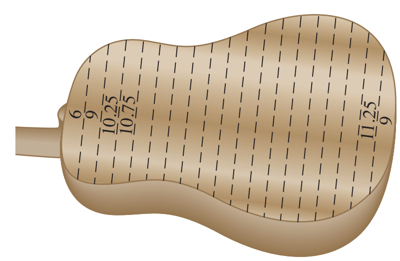


FIGURE 18 Back of guitar.

Exercises 57 - 58 use the notation and results of Exercises ??-?? of Section 3.4. For a given country, $F(r)$ is the fraction of total income that goes to the bottom rth fraction of households. The graph of $y = F(r)$ is called the Lorenz curve.

57. NEW Let A be the area between $y = r$ and $y = F(r)$ over the interval $[0, 1]$ (the shaded area in Figure 19). The **Gini index** is the ratio $G = A/B$ where B is the area under $y = r$ over $[0, 1]$.

(a) Show that $G = 2\int_0^1 (r - F(r))\, dr$

(b) Calculate G if $F(r) = \begin{cases} \frac{1}{3}r & \text{for } 0 \le r \le \frac{1}{2} \\ \frac{5}{3}r - \frac{2}{3} & \text{for } \frac{1}{2} \le r \le 1 \end{cases}$

(c) The Gini index is a measure of income distribution, with a lower value indicating more equal distribution. By Exercise **??** (d) of Section 3.4, all households have the same income if and only if $F(r) = r$. Calculate G in this case. What is G if all of the income goes to one household? Hint: in this extreme case, $F(r) = 0$ for $0 \le r < 1$.

58. NEW Calculate the Gini index of the U.S. in the year 2001, assuming that the graph of $F(r)$ in Figure 19 consists of straight line segments joined at the data points in the following table.

r	0	0.2	0.4	0.6	0.8	1
$F(r)$	0	0.035	0.123	0.269	0.499	1

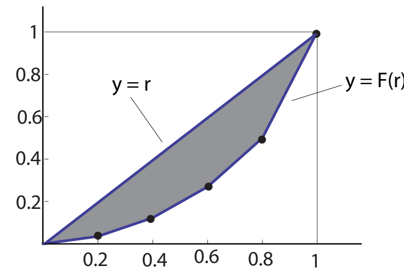


FIGURE 19 Lorenz curve for U.S. in 2001

Further Insights and Challenges

59. Find the line $y = mx$ that divides the area under the curve $y = x(1 - x)$ over $[0, 1]$ into two regions of equal area.

60. CAS Let c be the number such that the area under $y = \sin x$ over $[0, \pi]$ is divided in half by the line $y = cx$ (Figure 20). Find an equation for c and solve this equation *numerically* using a computer algebra system.

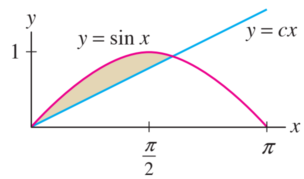


FIGURE 20

61. Explain geometrically (without calculation) why the following holds for any $n > 0$:

$$\int_0^1 x^n\,dx + \int_0^1 x^{1/n}\,dx = 1$$

62. Let $f(x)$ be a strictly increasing function with inverse $g(x)$. Explain the equality geometrically:

$$\int_0^a f(x)\,dx + \int_{f(0)}^{f(a)} g(x)\,dx = af(a)$$

6.2 Setting Up Integrals: Volume, Density, Average Value

In this section, we use the integral to compute quantities such as volume, total mass, and fluid flow. The common thread in these diverse applications is that we approximate the relevant quantity by a Riemann sum and then pass to the limit to obtain an exact value.

Volume

The term “solid” or “solid body” refers to a solid three-dimensional object.

We begin by showing how integration can be used to compute the **volume** of a solid body. Before proceeding, let’s recall that the volume of a *right cylinder* (Figure 1) is Ah, where A is the area of the base and h is the height, measured perpendicular to the base. Here we use the term “cylinder” in a general sense; the base does not have to be circular.

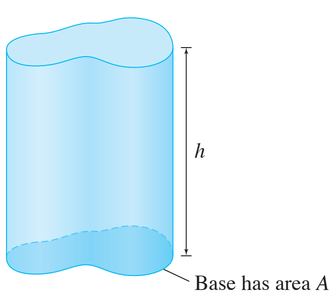


FIGURE 1 The volume of a right cylinder is Ah.

Now let V be the volume of a solid body that extends from height $y = a$ to $y = b$ along the y-axis as in Figure 2. The intersection of the solid with the horizontal plane at height y is called the **horizontal cross section** at height y. Let $A(y)$ be its area.

To compute V, we divide the solid into N horizontal slices of thickness $\Delta y = \dfrac{b-a}{N}$. The ith slice extends from y_{i-1} to y_i, where $y_i = a + i\,\Delta y$. Let V_i be the volume of the slice. If N is very large, then Δy is very small and the slices are very thin. In this case, the ith slice is nearly a right cylinder of base $A(y_{i-1})$ and height Δy, and therefore $V_i \approx A(y_{i-1})\Delta y$. Summing up, we obtain

$$V = \sum_{i=1}^{N} V_i \approx \sum_{i=1}^{N} A(y_{i-1})\Delta y$$

The sum on the right is a left-endpoint approximation to the integral $\displaystyle\int_a^b A(y)\,dy$. If we assume that $A(y)$ is a continuous function, then the approximation improves in accuracy and converges to the integral as $N \to \infty$. We conclude that *the volume of the solid is equal to the integral of its cross-sectional area.*

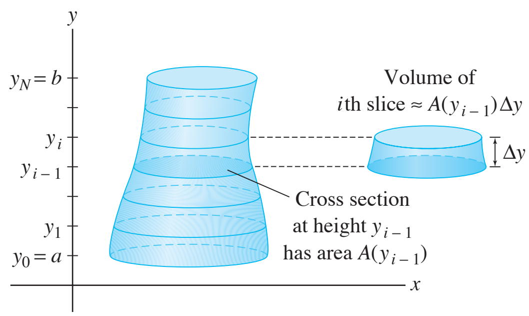


FIGURE 2 Divide the solid into thin horizontal slices. Each slice is nearly a right cylinder whose volume can be approximated as area times height.

Volume as the Integral of Cross-Sectional Area Suppose that a solid body extends from height $y = a$ to $y = b$. Let $A(y)$ be the area of the horizontal cross section at height y. Then

$$\text{Volume of the solid body} = \int_a^b A(y)\, dy \tag{1}$$

EXAMPLE 1 **Volume of a Pyramid: Horizontal Cross Sections** Calculate the volume V of a pyramid of height 12 m whose base is a square of side 4 m.

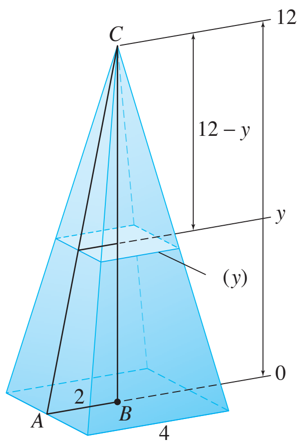


FIGURE 3 A horizontal cross section of the pyramid is a square.

Solution To use Eq. (1), we need a formula for $A(y)$.

***Step 1.* Find a formula for $A(y)$.**

We see in Figure 3 that the horizontal cross section at height y is a square. Let $\ell(y)$ be the length of its sides. We apply the law of similar triangles to $\triangle ABC$ and the triangle of height $12 - y$ whose base of length $\frac{1}{2}\ell(y)$ lies on the cross section:

$$\frac{\text{Base}}{\text{Height}} = \frac{2}{12} = \frac{\frac{1}{2}\ell(y)}{12 - y} \quad \Rightarrow \quad 2(12 - y) = 6\ell(y)$$

We find that $\ell(y) = \frac{1}{3}(12 - y)$ and therefore $A(y) = \ell(y)^2 = \frac{1}{9}(12 - y)^2$.

***Step 2.* Compute V as the integral of $A(y)$.**

$$V = \int_0^{12} A(y)\, dy = \int_0^{12} \frac{1}{9}(12 - y)^2 dy = -\frac{1}{27}(12 - y)^3 \Big|_0^{12} = 64 \text{ m}^3$$

Note that we would obtain this same result using the formula $V = \frac{1}{3}Ah$ for the volume of a pyramid of base A and height h, since $\frac{1}{3}Ah = \frac{1}{3}(4^2)(12) = 64$. ■

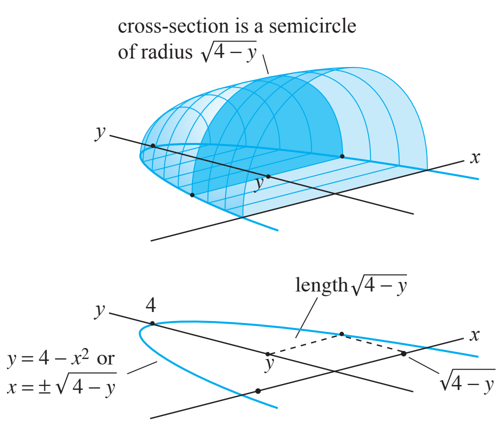


FIGURE 4 A solid whose base is the region between $y = 4 - x^2$ and the x-axis.

EXAMPLE 2 The base of a solid is the region between the x-axis and the inverted parabola $y = 4 - x^2$. The vertical cross sections of the solid perpendicular to the y-axis are semicircles (Figure 4). Compute the volume of the solid.

Solution First, we find a formula for the cross section at height y. Note that $y = 4 - x^2$ can be written $x = \pm\sqrt{4 - y}$. We see in Figure 4 that the cross section at height y is a semicircle of radius $r = \sqrt{4 - y}$. The area of the semicircle is $A(y) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}(4 - y)$, and the volume of the solid is

$$V = \int_0^4 A(y)\, dy = \frac{\pi}{2}\int_0^4 (4 - y)\, dy = \frac{\pi}{2}\left(4y - \frac{1}{2}y^2\right)\Big|_0^4 = 4\pi$$ ■

The volume of a solid body may be computed using vertical rather than horizontal cross sections. We then obtain an integral with respect to x rather than y.

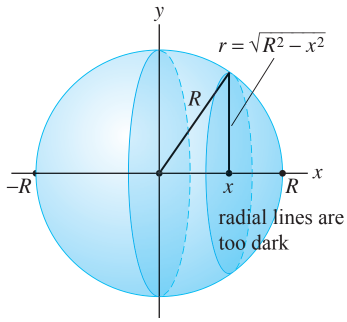


FIGURE 5 Vertical cross section is a circle of radius $\sqrt{R^2 - x^2}$.

■ **EXAMPLE 3 Volume of a Sphere: Vertical Cross Sections** Compute the volume of a sphere of radius R as an integral of cross-sectional area.

Solution As we see in Figure 5, the vertical cross section of the sphere at x is a circle whose radius r satisfies $x^2 + r^2 = R^2$ or $r = \sqrt{R^2 - x^2}$. The area of the cross section is $A(x) = \pi r^2 = \pi(R^2 - x^2)$. Therefore, the volume of the sphere is

$$\int_{-R}^{R} \pi(R^2 - x^2)\,dx = \pi\left(R^2 x - \frac{x^3}{3}\right)\Big|_{-R}^{R} = 2\left(\pi R^3 - \pi\frac{R^3}{3}\right) = \frac{4}{3}\pi R^3 \quad ■$$

Density

Next, we show how an integral may be used to compute the total mass of an object given its mass density. Consider a rod of length ℓ. The rod's **linear mass density** ρ is defined as the mass per unit length. If the density ρ is constant, then

$$\text{Total mass} = \text{linear mass density} \times \text{length} = \rho \cdot \ell \qquad \boxed{2}$$

For example, if $\ell = 10$ cm and $\rho = 9$ g/cm, then the total mass is $\rho\ell = 9 \cdot 10 = 90$ g.

Integration is needed to compute total mass when the density is not constant. Consider a rod extending along the x-axis from $x = a$ to $x = b$ whose density $\rho(x)$ depends on x, as in Figure 6. To compute the total mass M, we decompose the rod into N small segments of length $\Delta x = \frac{b-a}{N}$. Then $M = \sum_{i=1}^{N} M_i$, where M_i is the mass of the ith segment. Although Eq. (2) cannot be used when $\rho(x)$ is not constant, we can argue that if Δx is small, then $\rho(x)$ is nearly constant along the ith segment. Therefore, if the ith segment extends from x_{i-1} to x_i and if c_i is any sample point in $[x_{i-1}, x_i]$, then $M_i \approx \rho(c_i)\Delta x$ and

$$\text{Total mass } M = \sum_{i=1}^{N} M_i \approx \sum_{i=1}^{N} \rho(c_i)\Delta x$$

Do you see the similarity in the way we use thin slices to compute volume and small pieces to compute total mass?

As $N \to \infty$, the accuracy of the approximation improves. However, the sum on the right is a Riemann sum whose value approaches $\int_a^b \rho(x)\,dx$. We conclude that *the total mass of a rod is equal to the integral of its linear mass density*:

$$\boxed{\text{Total mass } M = \int_a^b \rho(x)\,dx} \qquad \boxed{3}$$

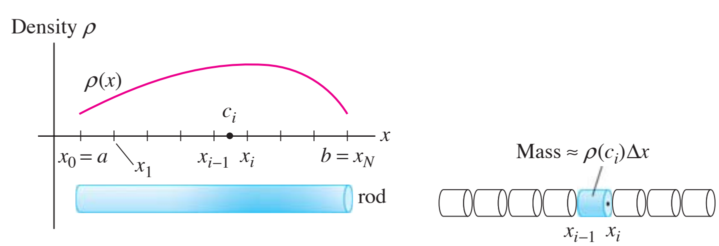


FIGURE 6 The total mass of the rod is equal to the area under the graph of mass density ρ.

EXAMPLE 4 **Total Mass** Find the total mass of a 2-m rod of linear density $\rho(x) = 1 + x(2 - x)$ kg/m, where x is the distance from one end of the rod.

Solution The total mass is

$$\int_0^2 \rho(x)\,dx = \int_0^2 \big(1 + x(2-x)\big)\,dx = \left(x + x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{10}{3}\text{ kg}$$ ■

In some situations, density is a function of distance to the origin. For example, in the study of urban populations, it might be assumed that the population density $\rho(r)$ (in people per square km) depends only on the distance r from the center of a city. Such a density function is called a **radial density function.**

More general density functions depend on two variables $\rho(x, y)$. In this case, total mass or population is computed using double integration, a topic in multivariable calculus.

We now derive a formula for the total population P within a radius R of the city center, assuming that the population density $\rho(r)$ is a radial density function. To compute P, it makes sense to divide the circle of radius R into N thin rings of equal width $\Delta r = R/N$ as in Figure 7.

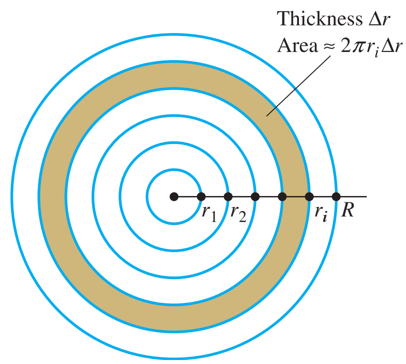


FIGURE 7 Dividing the circle of radius R into N thin rings of thickness $\Delta r = \dfrac{R}{N}$.

Let P_i be the population within the ith ring, so that $P = \sum_{i=1}^{N} P_i$. If the outer radius of the ith ring is r_i, then circumference is $2\pi r_i$, and if Δr is small, the area of this ring is *approximately* $2\pi r_i \Delta r$ (outer circumference times thickness). Furthermore, the population density within the thin ring is nearly constant with value $\rho(r_i)$. With these approximations,

$$P_i \approx \underbrace{2\pi r_i \Delta r}_{\text{Area of ring}} \times \underbrace{\rho(r_i)}_{\text{Population density}} = 2\pi r_i \rho(r_i)\Delta r$$

$$P = \sum_{i=1}^{N} P_i \approx 2\pi \sum_{i=1}^{N} r_i \rho(r_i)\Delta r$$

This last sum is a right-endpoint approximation to the integral $2\pi \int_0^R r\rho(r)\,dr$. As N tends to ∞, the approximation improves in accuracy and the sum converges to the integral. We conclude that for a population with a radial density function $\rho(r)$,

Remember that for a radial density function, the total population is obtained by integrating $2\pi r\rho(r)$ rather than $\rho(r)$.

$$\text{Population } P \text{ within a radius } R = 2\pi \int_0^R r\rho(r)\,dr \qquad \boxed{4}$$

EXAMPLE 5 Computing Total Population from Population Density The density function for the population in a certain city is $\rho(r) = 15(1 + r^2)^{-1/2}$, where r is the distance from the center in kilometers and ρ has units of thousands per square kilometer. How many people live within a 30-km radius of the city center?

Solution The population (in thousands) within a 30-km radius is

$$2\pi \int_0^{30} r\big(15(1+r^2)^{-1/2}\big)\, dr = 2\pi(15) \int_0^{30} \frac{r}{(1+r^2)^{1/2}}\, dr$$

To evaluate the integral, use the substitution $u = 1 + r^2$, $du = 2r\,dr$. The limits of integration become $u(0) = 1$ and $u(30) = 901$, and we obtain

$$30\pi \int_1^{901} u^{-1/2}\,(\frac{1}{2})du = 15\pi\, u^{1/2}\Big|_1^{901} \approx 1{,}367 \text{ thousand}$$

In other words, the population is nearly 1.4 million people. ■

Flow Rate

When liquid flows through a tube, the **flow rate** Q is the *volume per unit time* of fluid passing through the tube. The flow rate depends on the velocity of the fluid particles. If all particles of the liquid travel with the same velocity v (say, in units of centimeters per minute), then the flow rate through a tube of radius R is

$$\underbrace{\text{Flow rate } Q}_{\text{Volume per unit time}} = \text{cross-sectional area} \times \text{velocity} = \pi R^2 v \text{ cm}^3/\text{min}$$

How do we obtain this formula? Fix an observation point P in the tube and ask the following question: Which liquid particles flow past P in a 1-min interval? The particles passing P during this minute are located at most v centimeters to the left of P since each particle travels v centimeters per minute (assuming the liquid flows from left to right). Therefore, the column of liquid flowing past P in a 1-min interval is a cylinder of radius R and length v, which has volume $\pi R^2 v$ (Figure 8).

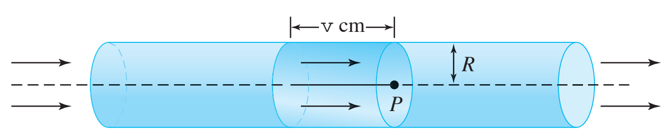


FIGURE 8 The column of fluid flowing past P in one unit of time is a cylinder of volume $\pi R^2 v$.

In reality, the particles of liquid do not all travel at the same velocity because of friction. However, for a slowly moving liquid, the flow is **laminar**, by which we mean that the velocity $v(r)$ depends only on the distance r from the center of the tube. The particles traveling along the center of the tube travel most quickly and the velocity tapers off to zero near the walls of the tube (Figure 9).

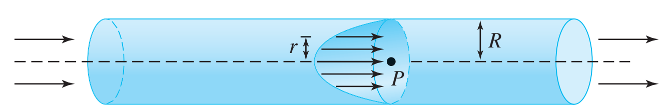


FIGURE 9 Laminar flow: Velocity of liquid increases toward the center of the tube.

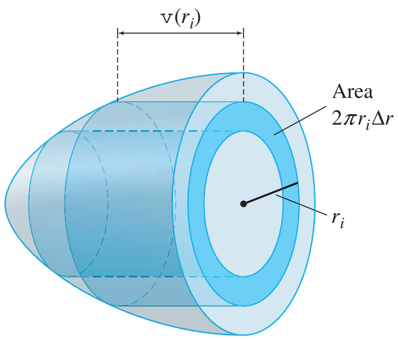


FIGURE 10 In a laminar flow, the fluid particles in a thin cylindrical shell all travel at nearly the same velocity.

If the flow is laminar, we can express the flow rate Q as an integral. The computation is similar to that of population with a radial density function. We divide the tube into N thin concentric cylindrical shells of width $\Delta r = R/N$ (Figure 10). The cross section of the cylindrical shell is a circular band. If r_i is the outer radius of the ith shell, then the area of this cross section is approximately $2\pi r_i \Delta r$. Furthermore, the fluid velocity within a shell is nearly constant with value $v(r_i)$, so we can approximate flow rate Q_i through the ith cylindrical shell by

$$Q_i \approx \text{cross-sectional area} \times \text{velocity} \approx 2\pi r_i \Delta r v(r_i)$$

We obtain

$$Q \approx \sum_{i=1}^{N} Q_i = 2\pi \sum_{i=1}^{N} r_i \, v(r_i) \Delta r$$

The sum on the right is a right-endpoint approximation to the integral $2\pi \int_0^R rv(r)\,dr$. Once again, we let N tend to ∞ to obtain the formula

$$\text{Flow rate } Q = 2\pi \int_0^R rv(r)\,dr \qquad \boxed{5}$$

The French physician Jean Poiseuille (1799–1869) discovered the law of laminar flow that cardiologists use to study blood flow in humans. Poiseuille's Law highlights the danger of cholesterol buildup in blood vessels: The flow rate through a blood vessel of radius R is proportional to R^4, so if R is reduced by one-half, the flow is reduced by a factor of 16.

EXAMPLE 6 Poiseuille's Law of Laminar Flow According to Poiseuille's Law, the velocity of blood flowing in a blood vessel of radius R cm is $v(r) = k(R^2 - r^2)$, where r is the distance from the center of the vessel (in centimeters) and k is a constant. Calculate the flow rate Q as function of R, assuming that $k = 0.5$ (cm-s)$^{-1}$.

Solution By Eq. (5),

$$Q = 2\pi \int_0^R (0.5) r(R^2 - r^2)\,dr = \pi \left(R^2 \frac{r^2}{2} - \frac{r^4}{4} \right) \Bigg|_0^R = \frac{\pi}{4} R^4 \text{ cm}^3/\text{s}$$

This shows that the flow rate is proportional to R^4 (this is true for any value of k). ■

Average Value

As a final example, we discuss the *average value* of a function. Recall that the average of N numbers $a_1, a_2, \ldots, a_N$ is the sum divided by N:

$$\frac{a_1 + a_2 + \cdots + a_N}{N} = \frac{1}{N} \sum_{j=1}^{N} a_j$$

For example, the average of 18, 25, 22, and 31, is $\frac{1}{4}(18 + 25 + 22 + 31) = 24$.

We cannot define the average value of a function $f(x)$ on an interval $[a, b]$ as a sum because there are infinitely many values of x to consider. But notice that the right-endpoint approximation R_N may be interpreted in terms of an average value (Figure 11):

$$R_N = \frac{b-a}{N} \big(f(x_1) + f(x_2) + \cdots + f(x_N) \big)$$

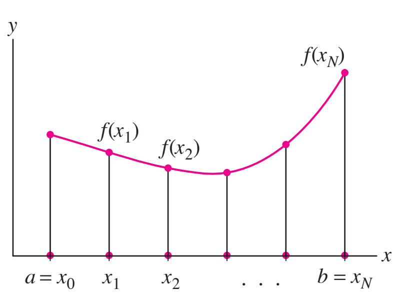


FIGURE 11 $\dfrac{1}{b-a} R_N$ is equal to the average of the values of $f(x)$ at the points $x_1, x_2, \ldots, x_N$.

where $x_i = a + i\left(\frac{b-a}{N}\right)$. Dividing by $(b - a)$, we obtain the average of the equally spaced function values $f(x_i)$:

$$\frac{1}{b-a} R_N = \underbrace{\frac{f(x_1) + f(x_2) + \cdots + f(x_N)}{N}}_{\text{Average of the function values}}$$

If N is large, it is reasonable to think of this quantity as an *approximation* to the average of $f(x)$ on $[a, b]$. Therefore, we define the average value itself as the limit:

$$\text{Average value} = \lim_{N\to\infty} \frac{1}{b-a} R_N(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The average value is also called the **mean value**.

DEFINITION Average Value The **average value** of an integrable function $f(x)$ on $[a, b]$ is the quantity

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \boxed{6}$$

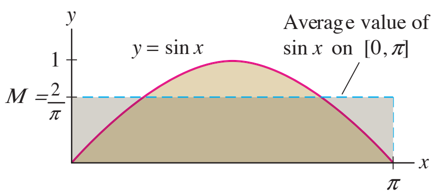


FIGURE 12 The area under the graph is equal to the area of the rectangle of height M, where M is the average value.

■ **EXAMPLE 7** Find the average value of $f(x) = \sin x$ on **(a)** $[0, \pi]$ and **(b)** $[0, 2\pi]$.

Solution

(a) The average value of $\sin x$ on $[0, \pi]$ is

$$\frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = \frac{1}{\pi}\big(-(-1) - (-1)\big) = \frac{2}{\pi} \approx 0.637$$

This answer is reasonable because $\sin x$ varies from 0 to 1 on the interval $[0, \pi]$ and the average 0.637 lies somewhere between the two extremes (Figure 12).

(b) The average value over $[0, 2\pi]$ is

$$\frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = -\frac{1}{2\pi}\cos x\Big|_0^{2\pi} = -\frac{1}{2\pi}\big(1 - (1)\big) = 0$$

This answer is also reasonable: The positive and negative values of $\sin x$ on $[0, 2\pi]$ cancel each other out, yielding an average of zero. ■

GRAPHICAL INSIGHT The average value M of a function $f(x)$ on $[a, b]$ is the average height of its graph over $[a, b]$ (Figure 12). Furthermore, the (signed) area of the rectangle of height M over $[a, b]$, $M(b-a)$, is equal to $\int_a^b f(x)\,dx$.

■ **EXAMPLE 8 Vertical Jump of a Bushbaby** The bushbaby (Galago Senegalensis) is a small primate with remarkable jumping ability. Find the average speed during a jump if the initial vertical velocity is $v_0 = 600$ cm/s. Use Galileo's formula for the height $h(t) = v_0 t - 490t^2$ (in centimeters).

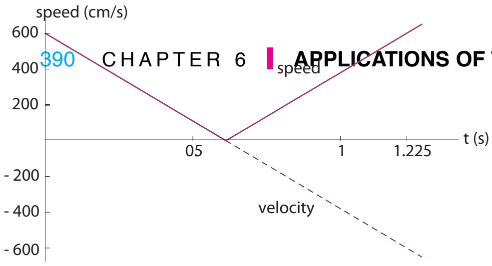


FIGURE 14 Graph of speed $|h'(t)| = |600 - 980t|$.

Solution First, determine the time interval of the jump. Since $v_0 = 600$, the bushbaby's height at time t is $h(t) = 600t - 490t^2 = t(600 - 490t)$. We have $h(0) = h(600/490) = 0$, so the jump begins at $t = 0$ s and ends at $t = 600/490 \approx 1.225$ s.

The bushbaby's speed is $|h'(t)| = |600 - 980t|$. We calculate the average speed as a sum of two integrals since $h'(t)$ is positive on $[0, 0.6125]$ and negative on $[0.6125, 1.225]$ (Figure 14):

$$\frac{1}{1.225 - 0}\int_0^{1.225} |600 - 980t|\,dt = \frac{1}{0.225}\left(\int_0^{0.6125}(600 - 980t)\,dt + \int_{0.6125}^{1.225}(980t - 600)\,dt\right)$$

$$= \frac{600t - 490t^2}{0.225}\Big|_0^{0.6125} + \frac{490t^2 - 600t}{0.225}\Big|_{0.6125}^{1.225}$$

$$\approx 150 + 150 = 300 \text{ cm/s}$$

■

There is an important difference between the average of a list of numbers and the average value of a continuous function. If the average score on an exam is 84, then 84 lies between the highest and lowest scores, but it may happen that no student received a score of 84. By contrast, our next result, called the Mean Value Theorem (MVT) for Integrals, asserts that a continuous function always takes on its average value at some point in the interval (Figure 15).

THEOREM 1 Mean Value Theorem for Integrals If $f(x)$ is continuous on $[a, b]$, then there exists a value $c \in [a, b]$ such that

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

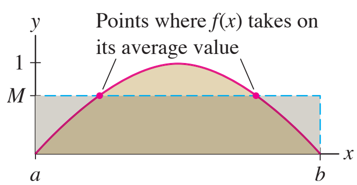


FIGURE 15 The function $f(x)$ takes on its average value M at the points where the upper edge of the rectangle intersects the graph.

Notice how the proof of Theorem 1 uses important parts of the theory we have developed so far: the existence of extreme values on a closed interval, the IVT, and the basic property that if $f(x) \le g(x)$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Proof Because $f(x)$ is continuous and $[a, b]$ is closed, there exist points $c_{\min}$ and $c_{\max}$ in $[a, b]$ such that $f(c_{\min})$ and $f(c_{\max})$ are the minimum and maximum values of $f(x)$ on $[a, b]$. Thus, $f(c_{\min}) \le f(x) \le f(c_{\max})$ for all $x \in [a, b]$ and therefore,

$$\int_a^b f(c_{\min})\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(c_{\max})\,dx$$

$$f(c_{\min})(b - a) \le \int_a^b f(x)\,dx \le f(c_{\max})(b - a)$$

Now divide by $(b - a)$:

$$f(c_{\min}) \le \underbrace{\frac{1}{b - a}\int_a^b f(x)\,dx}_{\text{Average value } M} \le f(c_{\max}) \qquad \boxed{7}$$

The expression in the middle is the average value M of $f(x)$ on $[a, b]$. We see that M lies between the min and max of $f(x)$ on $[a, b]$. Because $f(x)$ is continuous, the Intermediate Value Theorem (IVT) guarantees that $f(c) = M$ for some $c \in [a, b]$ as claimed. ■

6.2 SUMMARY

- The volume V of a solid body is equal to the integral of the area of the horizontal (or vertical) cross sections $A(y)$:

$$V = \int_a^b A(y)\,dy$$

- *Linear mass density* $\rho(x)$ is defined as mass per unit length. If a rod (or other object) with density ρ extends from $x = a$ to $x = b$, then its total mass is $M = \int_a^b \rho(x)\,dx$.
- If the density function $\rho(r)$ depends only on the distance r from the origin (*radial density function*), then the total amount (of population, mass, etc.) within a radius R of the center is equal to $2\pi \int_0^R r\rho(r)\,dr$.
- The volume of fluid passing through a tube of radius R per unit time is called the *flow rate* Q. The flow is *laminar* if the velocity $v(r)$ of a fluid particle depends only on its distance r from the center of the tube. For a laminar flow, $Q = 2\pi \int_0^R rv(r)\,dr$.
- The *average* (or *mean*) value on $[a, b]$: $M = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.
- The MVT for Integrals: If $f(x)$ is continuous on $[a, b]$ with average value M, then $f(c) = M$ for some $c \in [a, b]$.

6.2 EXERCISES

Preliminary Questions

1. What is the average value of $f(x)$ on $[1, 4]$ if the area between the graph of $f(x)$ and the x-axis is equal to 9?

2. Find the volume of a solid extending from $y = 2$ to $y = 5$ if the cross section at y has area $A(y) = 5$ for all y.

3. Describe the horizontal cross sections of an ice cream cone and the vertical cross sections of a football (when it is held horizontally).

4. What is the formula for the total population within a circle of radius R around a city center if the population has a radial function?

5. What is the definition of flow rate?

6. Which assumption about fluid velocity did we use to compute the flow rate as an integral?

Exercises

1. Let V be the volume of a pyramid of height 20 whose base is a square of side 8.

(a) Use similar triangles as in Example 1 to find the area of the horizontal cross section at a height y.

(b) Calculate V by integrating the cross-sectional area.

2. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4 (Figure 16).

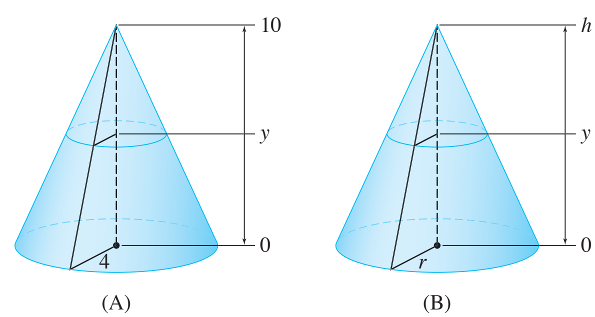


FIGURE 16 Right circular cones.

(a) Use similar triangles to find the area of a horizontal cross section at a height y.

(b) Calculate V by integrating the cross-sectional area.

3. Use the method of Exercise 2 to find the formula for the volume of a right circular cone of height h whose base is a circle of radius r (Figure 16).

4. Calculate the volume of the ramp in Figure 17 in three ways by integrating the area of the cross sections:

(a) Perpendicular to the x-axis (rectangles)

(b) Perpendicular to the y-axis (triangles)

(c) Perpendicular to the z-axis (rectangles)

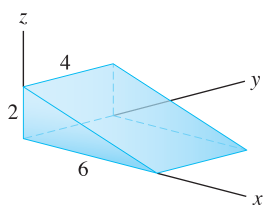


FIGURE 17 Ramp of length 6, width 4, and height 2.

5. Find the volume of liquid needed to fill a sphere of radius R to height h (Figure 18).

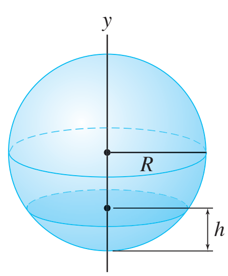


FIGURE 18 Sphere filled with liquid to height h.

6. Find the volume of the wedge in Figure 19(A) by integrating the area of vertical cross sections.

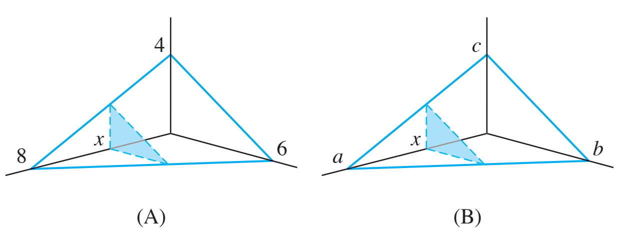


FIGURE 19

7. Derive a formula for the volume of the wedge in Figure 19(B) in terms of the constants a, b, and c.

8. Let B be the solid whose base is the unit circle $x^2 + y^2 = 1$ and whose vertical cross sections perpendicular to the x-axis are equilateral triangles. Show that the vertical cross sections have area $A(x) = \sqrt{3}(1 - x^2)$ and compute the volume of B.

In Exercises 9–14, find the volume of the solid with given base and cross sections.

9. The base is the unit circle $x^2 + y^2 = 1$ and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

10. The base is the triangle enclosed by $x + y = 1$, the x-axis, and the y-axis. The cross sections perpendicular to the y-axis are semicircles.

11. The base is the semicircle $y = \sqrt{9 - x^2}$, where $-3 \le x \le 3$. The cross sections perpendicular to the x-axis are squares.

12. The base is a square, one of whose sides is the interval $[0, \ell]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

13. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

14. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are rectangles of height y^3.

15. Find the volume of the solid whose base is the region $|x| + |y| \le 1$ and whose vertical cross sections perpendicular to the y-axis are semicircles (with diameter along the base).

16. Show that the volume of a pyramid of height h whose base is an equilateral triangle of side s is equal to $\frac{\sqrt{3}}{12}hs^2$.

17. Find the volume V of a *regular* tetrahedron whose face is an equilateral triangle of side s (Figure 20).

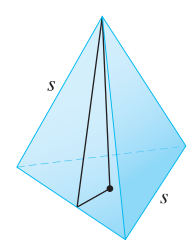


FIGURE 20 Regular tetrahedron.

18. The area of an ellipse is πab, where a and b are the lengths of the semimajor and semiminor axes (Figure 21). Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.

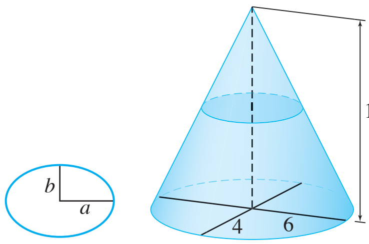


FIGURE 21

19. A frustum of a pyramid is a pyramid with its top cut off [Figure 22(A)]. Let V be the volume of a frustum of height h whose base is a square of side a and top is a square of side b with $a > b \geq 0$.

(a) Show that if the frustum were continued to a full pyramid, it would have height $\dfrac{ha}{a-b}$ [Figure 22(B)].

(b) Show that the cross section at height x is a square of side $(1/h)(a(h-x)+bx)$.

(c) Show that $V = \frac{1}{3}h(a^2+ab+b^2)$. A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4,000 years ago.

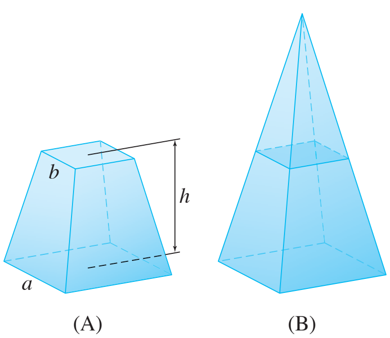


FIGURE 22

20. A plane inclined at an angle of 45° passes through a diameter of the base of a cylinder of radius r. Find the volume of the region within the cylinder and below the plane (Figure 23).

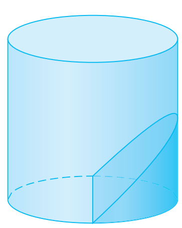


FIGURE 23

21. Figure 24 shows the solid S obtained by intersecting two cylinders of radius r whose axes are perpendicular.

(a) The horizontal cross section of each cylinder at distance y from the central axis is a rectangular strip. Find the strip's width.

(b) Find the area of the horizontal cross section of S at distance y.

(c) Find the volume of S as a function of r.

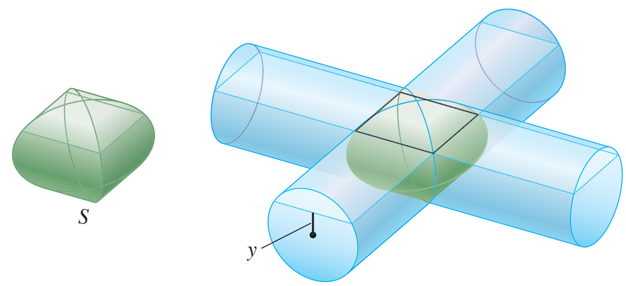


FIGURE 24 Intersection of two cylinders intersecting at right angles.

22. Let S be the solid obtained by intersecting two cylinders of radius r whose axes intersect at an angle θ. Find the volume of S as a function of r and θ.

23. Calculate the volume of a cylinder inclined at an angle $\theta = 30°$ whose height is 10 and whose base is a circle of radius 4 (Figure 25).

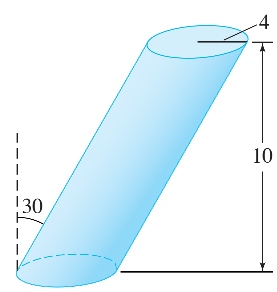


FIGURE 25 Cylinder inclined at an angle $\theta = 30°$.

24. Find the total mass of a 1-m rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \leq x \leq 1$.

25. Find the total mass of a 2-m rod whose linear density function is $\rho(x) = 1 + 0.5\sin(\pi x)$ kg/m for $0 \leq x \leq 2$.

26. A mineral deposit along a strip of length 6 cm has density $s(x) = 0.01x(6-x)$ g/cm for $0 \leq x \leq 6$. Calculate the total mass of the deposit.

27. Calculate the population within a 10-mile radius of the city center if the radial population density is $\rho(r) = 4(1+r^2)^{1/3}$ (in thousands per square mile).

28. Odzala National Park in the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1+r^2)^{-2}$ gorillas per square kilometer, where r is the distance from a large grassy clearing with a source of food and water. Calculate the number of gorillas within a 5-km radius of the clearing.

29. Table 1 lists the population density (in people per squared kilometer) as a function of distance r (in kilometers) from the center of a rural town. Estimate the total population within a 2-km radius of the center by taking the average of the left- and right-endpoint approximations.

TABLE 1 Population Density

r	$\rho(r)$	r	$\rho(r)$
0.0	125.0	1.2	37.6
0.2	102.3	1.4	30.8
0.4	83.8	1.6	25.2
0.6	68.6	1.8	20.7
0.8	56.2	2.0	16.9
1.0	46.0		

30. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.

31. The density of deer in a forest is the radial function $\rho(r) = 150(r^2+2)^{-2}$ deer per km^2, where r is the distance (in kilometers) to a small meadow. Calculate the number of deer in the region $2 \le r \le 5$ km.

32. Show that a circular plate of radius 2 cm with radial mass density $\rho(r) = \dfrac{4}{r}$ g/cm has finite total mass, even though the density becomes infinite at the origin.

33. Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance r cm from the center is $v(r) = 16 - r^2$ cm/s.

34. Let $v(r)$ be the velocity of blood in an arterial capillary of radius $R = 4 \times 10^{-5}$ m. Use Poiseuille's Law (Example 6) with $k = 10^6$ (m-s)$^{-1}$ to determine the velocity at the center of the capillary and the flow rate (use correct units).

35. A solid rod of radius 1 cm is placed in a pipe of radius 3 cm so that their axes are aligned. Water flows through the pipe and around the rod. Find the flow rate if the velocity of the water is given by the radial function $v(r) = 0.5(r-1)(3-r)$ cm/s.

36. MODIFIED The areas of cross sections to Lake Nogebow at 5 meter intervals are given in the table below. Figure 26 shows a contour map of the lake. Estimate the volume V of the lake by taking the average of the right- and left-endpoint approximations to the integral of cross-sectional area.

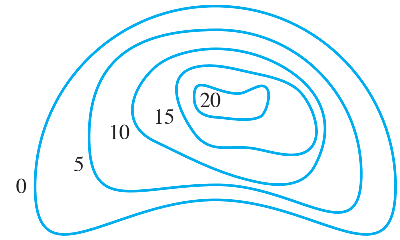


FIGURE 26 Depth contour map of Lake Nogebow.

Depth (m)	0	5	10	15	20
Area (million m^2)	2.1	1.5	1.1	0.835	0.217

In Exercises 37–46, calculate the average over the given interval.

37. $f(x) = x^3$, $[0, 1]$

38. $f(x) = x^3$, $[-1, 1]$

39. $f(x) = \cos x$, $[0, \frac{\pi}{2}]$

40. $f(x) = \sec^2 x$, $[0, \frac{\pi}{4}]$

41. $f(s) = s^{-2}$, $[2, 5]$

42. $f(x) = \dfrac{\sin(\pi/x)}{x^2}$, $[1, 2]$

43. $f(x) = 2x^3 - 3x^2$, $[-1, 3]$

44. $f(x) = x^n$, $[0, 1]$

45. $f(x) = \dfrac{1}{x^2+1}$, $[-1, 1]$

46. $f(x) = e^{-nx}$, $[-1, 1]$

47. Let M be the average value of $f(x) = x^3$ on $[0, A]$, where $A > 0$. Which theorem guarantees that $f(c) = M$ has a solution c in $[0, A]$? Find c.

48. CAS Let $f(x) = 2\sin x - x$. Use a computer algebra system to plot $f(x)$ and estimate:

(a) The positive root α of $f(x)$.

(b) The average value M of $f(x)$ on $[0, \alpha]$.

(c) A value $c \in [0, \alpha]$ such that $f(c) = M$.

49. Which of $f(x) = x\sin^2 x$ and $g(x) = x^2 \sin^2 x$ has a larger average value over $[0, 1]$? Over $[1, 2]$?

50. Show that the average value of $f(x) = \dfrac{\sin x}{x}$ over $[\frac{\pi}{2}, \pi]$ is less than 0.41. Sketch the graph if necessary.

51. Sketch the graph of a function $f(x)$ such that $f(x) \ge 0$ on $[0, 1]$ and $f(x) \le 0$ on $[1, 2]$, whose average on $[0, 2]$ is negative.

52. Find the average of $f(x) = ax + b$ over the interval $[-M, M]$, where a, b, and M are arbitrary constants.

53. The temperature $T(t)$ at time t (in hours) in an art museum varies according to $T(t) = 70 + 5\cos\left(\frac{\pi}{12}t\right)$. Find the average over the time periods $[0, 24]$ and $[2, 6]$.

54. A ball is thrown in the air vertically from ground level with initial velocity 64 ft/s. Find the average height of the ball over the time interval extending from the time of the ball's release to its return to ground level. Recall that the height at time t is $h(t) = 64t - 16t^2$.

55. What is the average area of the circles whose radii vary from 0 to 1?

56. An object with zero initial velocity accelerates at a constant rate of 10 m/s^2. Find its average velocity during the first 15 s.

57. The acceleration of a particle is $a(t) = t - t^3$ m/s^2 for $0 \le t \le 1$. Compute the average acceleration and average velocity over the time interval $[0, 1]$, assuming that the particle's initial velocity is zero.

58. Let M be the average value of $f(x) = x^4$ on $[0, 3]$. Find a value of c in $[0, 3]$ such that $f(c) = M$.

59. Let $f(x) = \sqrt{x}$. Find a value of c in $[4, 9]$ such that $f(c)$ is equal to the average of f on $[4, 9]$.

60. Give an example of a function (necessarily discontinuous) that does not satisfy the conclusion of the MVT for Integrals.

Further Insights and Challenges

61. An object is tossed in the air vertically from ground level with initial velocity v_0 ft/s at time $t = 0$. Find the average speed of the object over the time interval $[0, T]$, where T is the time the object returns to earth.

62. Review the MVT stated in Section 4.3 (Theorem 1, p. 230) and show how it can be used, together with the Fundamental Theorem of Calculus, to prove the MVT for integrals.

6.3 Volumes of Revolution

We use the terms "revolve" and "rotate" interchangeably.

A **solid of revolution** is a solid obtained by rotating a region in the plane about an axis. The sphere and right circular cone are familiar examples of such solids. Each of these is "swept out" as a plane region revolves around an axis (Figure 1).

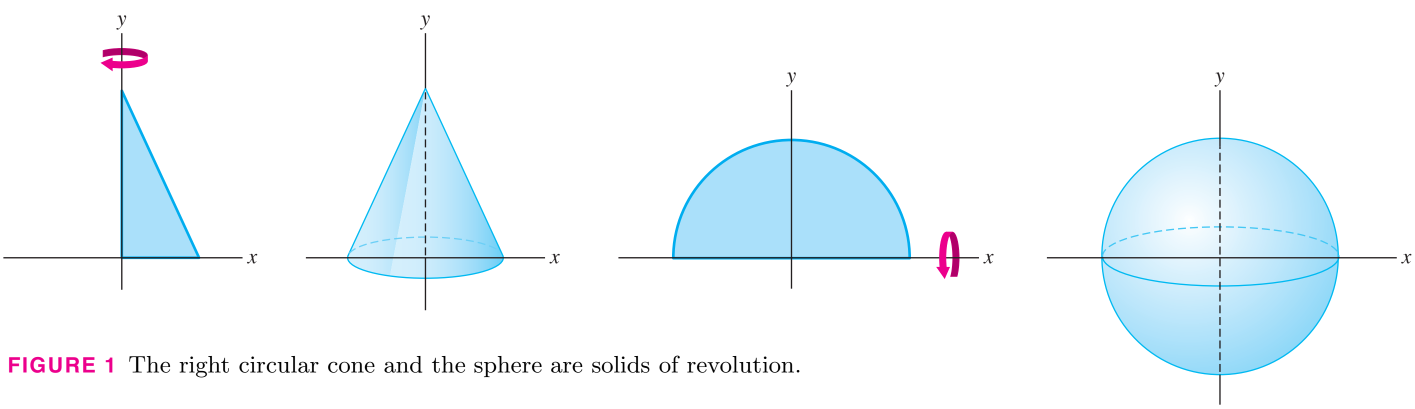


FIGURE 1 The right circular cone and the sphere are solids of revolution.

This method for computing the volume is often referred to as the "disk method" because the vertical slices of the solid are circular disks.

In general, if $f(x) \ge 0$ for $a \le x \le b$, then the region under the graph lies above the x-axis. Rotating this region around the x-axis produces a solid with a special feature: All vertical cross sections are circles (Figure 2). In fact, the vertical cross section at location x is a circle of radius $R = f(x)$ and has area

$$\text{Area of the vertical cross section} = \pi R^2 = \pi f(x)^2$$

As we saw in Section 6.2, the total volume V is equal to the integral of cross-sectional area. Therefore, $V = \int_a^b \pi f(x)^2\, dx$.

Volume of a Solid of Revolution: Disk Method If $f(x)$ is continuous and $f(x) \geq 0$ on $[a, b]$, then the volume V obtained by rotating the region under the graph about the x-axis is [with $R = f(x)$]:

$$V = \pi \int_a^b R^2\, dx = \pi \int_a^b f(x)^2\, dx \qquad \boxed{1}$$

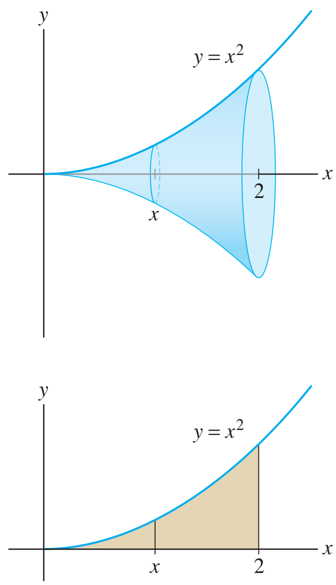


FIGURE 3 Region under $y = x^2$ rotated about the x-axis.

■ **EXAMPLE 1** Calculate the volume V of the solid obtained by rotating the region under $y = x^2$ about the x-axis for $0 \leq x \leq 2$.

Solution In this case, $f(x) = x^2$ (Figure 3), and by Eq. (1),

$$V = \pi \int_0^2 R^2\, dx = \pi \int_0^2 (x^2)^2\, dx = \pi \int_0^2 x^4\, dx = \pi \left.\frac{x^5}{5}\right|_0^2 = \pi \frac{2^5}{5} = \frac{32}{5}\pi \qquad ■$$

We now consider some variations on the formula for a volume of revolution. First, consider the region *between* two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x) \geq 0$ as in Figure 5 (A). When we rotate the region about the x-axis, segment $\overline{AB}$ sweeps out the **washer** shown in in Figure 5 (B). This washer (also called an annulus – see Figure 4) has outer radius $R_{\text{outer}} = f(x)$ and inner radius $R_{\text{inner}} = g(x)$. The area of the washer is $\pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2$ or $\pi(f(x)^2 - g(x)^2)$, and we obtain the volume of the solid of revolution as the integral of the cross-sectional area:

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx \qquad \boxed{2}$$

Keep in mind that $f(x)^2$ denotes the square $\big(f(x)\big)^2$ and should not be confused with $f(x^2)$.

■ **EXAMPLE 2 Rotating the Area Between Two Curves** Find the volume V of the solid obtained by rotating the region between $y = x^2 + 4$ and $y = 2$ about the x-axis for $1 \leq x \leq 3$.

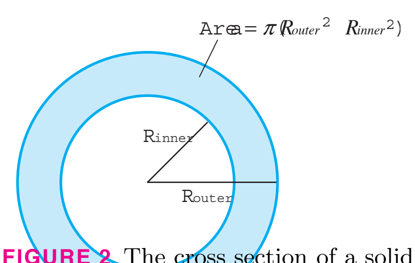
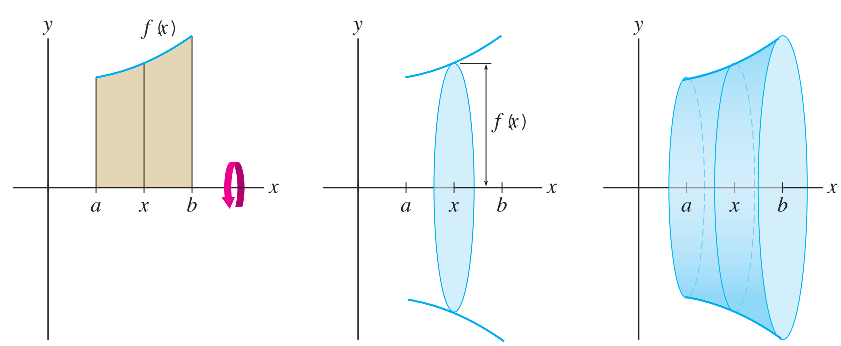


FIGURE 2 The cross section of a solid of revolution is a circle of radius $f(x)$ and area $\pi f(x)^2$.

FIGURE 4 The region between two concentric circles is called an "annulus," or more informally, a "washer."



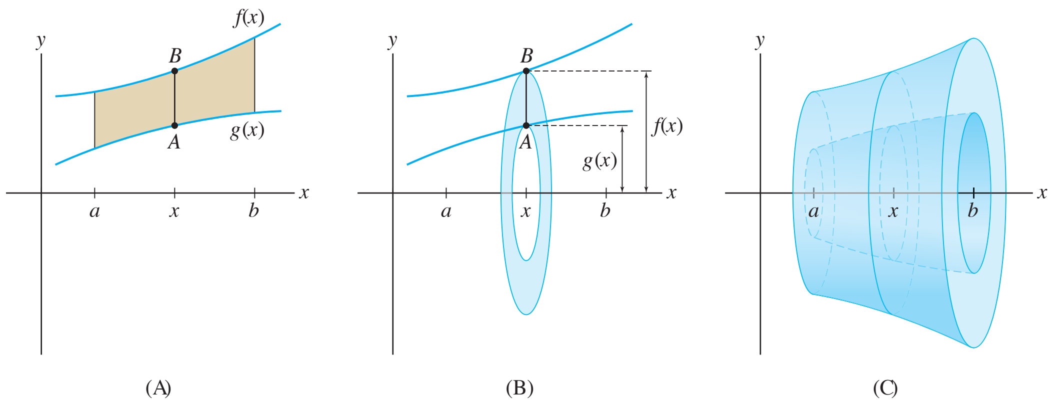


FIGURE 5 The vertical cross section is the washer generated when $\overline{AB}$ is rotated about the x-axis.

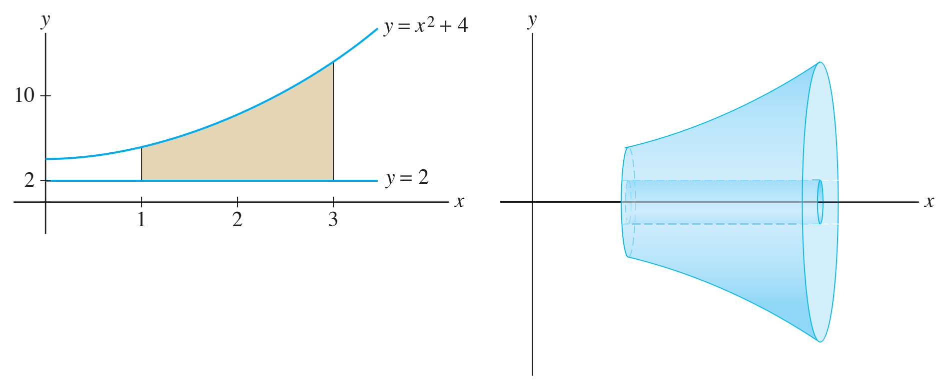


FIGURE 6 The area between $y = x^2 + 4$ and $y = 2$ over $[1, 3]$ rotated about the x-axis.

Solution The graph of $y = x^2 + 4$ lies above the graph of $y = 2$ (Figure 6). Therefore, $R_{\text{outer}} = x^2 + 4$ and $R_{\text{inner}} = 2$. By Eq. (2),

$$V = \pi \int_1^3 \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx = \pi \int_1^3 \left((x^2+4)^2 - 2^2\right) dx$$

$$= \pi \int_1^3 \left(x^4 + 8x^2 + 12\right) dx = \pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x\right)\Bigg|_1^3 = \frac{2{,}126}{15}\pi$$

Equation (2) can be modified to compute volumes of revolution about horizontal lines parallel to the x-axis.

EXAMPLE 3 **Revolving About a Horizontal Axis** Find the volume V of the "wedding band" in Figure 7(C), obtained by rotating the region between the graphs of $f(x) = x^2 + 2$ and $g(x) = 4 - x^2$ about the horizontal line $y = -3$.

Solution First, let's determine the points of intersection of the two graphs by solving

$$x^2 + 2 = 4 - x^2 \qquad \text{or} \qquad x^2 = 1$$

The graphs intersect at $x = \pm 1$. Figure 7(A) shows that the graph of $g(x) = 4 - x^2$ lies above the graph of $f(x) = x^2 + 2$ on $[-1, 1]$.

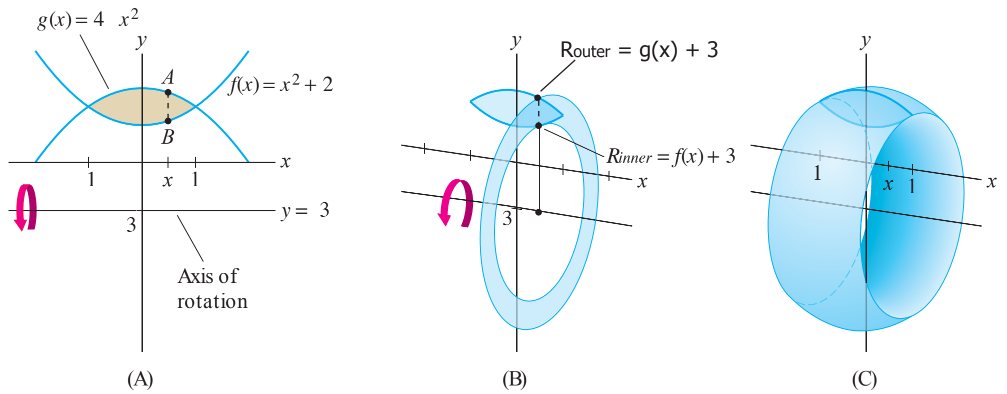


FIGURE 7 Segment $\overline{AB}$ generates a washer when rotated about the axis $y = -3$, but the inner and outer radii are 3 units longer.

Step 1. **Warmup.**

If we revolved about the x-axis, the volume would be given by Eq. (2):

$$V \text{ (about } x\text{-axis)} = \pi \int_{-1}^{1} \left(g(x)^2 - f(x)^2\right) dx = \pi \int_{-1}^{1} \left((4 - x^2)^2 - (x^2 + 2)^2\right) dx$$

Step 2. **Revolving about $y = -3$.**

When you set up the integral for a volume of revolution, visualize the cross sections. These cross sections are washers (or disks) whose inner and outer radii depend on the axis of rotation.

The formula is similar, but we must use the appropriate outer and inner radii. As we see in Figure 7(B), when we rotate about $y = -3$, $\overline{AB}$ generates a washer whose outer and inner radii are both 3 units longer:

- R_{outer} extends from $y = -3$ to $y = g(x)$, so $R_{\text{outer}} = g(x) - (-3) = 7 - x^2$
- R_{inner} extends from $y = -3$ to $y = f(x)$, so $R_{\text{inner}} = f(x) - (-3) = x^2 + 5$

The volume of revolution is obtained by integrating the area of this washer:

$$V \text{ (about } y = -3) = \pi \int_{-1}^{1} \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx = \pi \int_{-1}^{1} \left(\left(g(x) + 3\right)^2 - \left(f(x) + 3\right)^2\right) dx$$

$$= \pi \int_{-1}^{1} \left((7 - x^2)^2 - (x^2 + 5)^2\right) dx$$

$$= \pi \int_{-1}^{1} \left((49 - 14x^2 + x^4) - (x^4 + 10x^2 + 25)\right) dx$$

$$= \pi \int_{-1}^{1} (24 - 24x^2)\, dx = \pi(24x - 8x^3)\Big|_{-1}^{1} = 32\pi$$

■ **EXAMPLE 4** Find the volume of the solid obtained by rotating the region between the graph of $f(x) = 9 - x^2$ and the line $y = 12$ for $0 \le x \le 3$ about

(a) the line $y = 12$ **(b)** the line $y = 15$.

Solution To set up the integrals, we must first visualize whether the cross section is a disk or washer.

In Figure 8, the length of $\overline{AB}$ is $12 - f(x)$ rather than $f(x) - 12$ because the line $y = 12$ lies above the graph of $f(x)$.

(a) Figure 8(B) shows that $\overline{AB}$ rotated about $y = 12$ generates a disk of radius

$$R = \text{length of } \overline{AB} = 12 - f(x) = 12 - (9 - x^2) = 3 + x^2$$

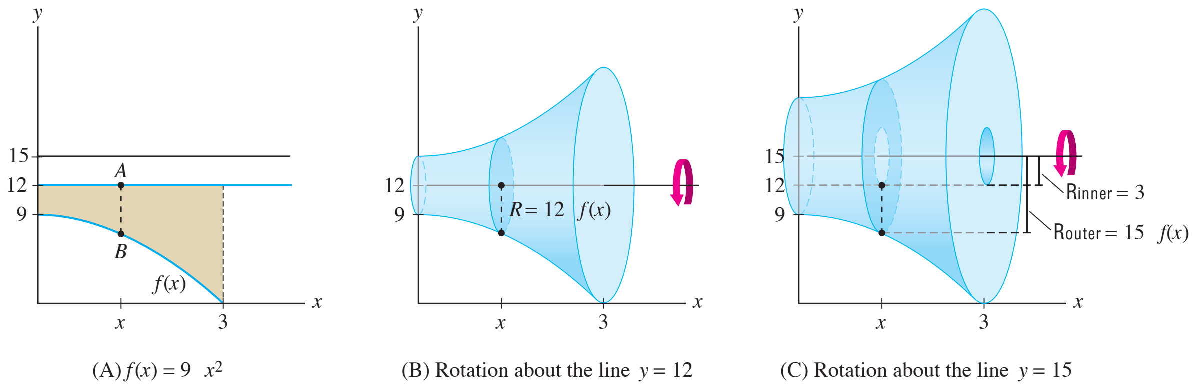


FIGURE 8 Segment $\overline{AB}$ generates a disk when rotated about $y = 12$, but it generates a washer when rotated about $y = 15$.

The volume of the solid of revolution about $y = 12$ is

$$\pi \int_0^3 R^2\,dx = \pi \int_0^3 (3 + x^2)^2\,dx = \pi \int_0^3 (9 + 6x^2 + x^4)\,dx$$
$$= \pi \left(9x + 2x^3 + \frac{1}{5}x^5 \right) \bigg|_0^3 = \frac{648}{5}\pi$$

(b) Figure 8(C) shows that $\overline{AB}$ rotated about $y = 15$ generates a washer. The outer radius of this washer is the distance from B to the line $y = 15$:

$$R_{\text{outer}} = 15 - f(x) = 15 - (9 - x^2) = 6 + x^2$$

The inner radius is $R_{\text{inner}} = 3$, so the volume of revolution about $y = 15$ is

$$\pi \int_0^3 (R_{\text{outer}}^2 - R_{\text{inner}}^2)\,dx = \pi \int_0^3 \left((6 + x^2)^2 - 3^2\right)dx = \pi \int_0^3 \left(36 + 12x^2 + x^4 - 9\right)dx$$
$$= \pi \left(27x + 4x^3 + \frac{1}{5}x^5 \right) \bigg|_0^3 = \frac{1{,}188}{5}\pi$$

We can use the disk and washer methods for solids of revolution about vertical axes provided that we describe the graph as a function of y rather than x.

EXAMPLE 5 **Revolving About a Vertical Axis** Find the volume of the solid obtained by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \le x \le 3$ about the vertical axis $x = -2$.

Solution Figure 9 shows that $\overline{AB}$ sweeps out a horizontal washer when rotated about the vertical line $x = -2$. We are going to integrate with respect to y, so we need the inner and outer radii of this washer as functions of y. Solving for x in $y = 9 - x^2$, we obtain $x^2 = 9 - y$ or $x = \sqrt{9 - y}$. Therefore,

$$R_{\text{outer}} = \sqrt{9 - y} + 2, \qquad R_{\text{inner}} = 2$$

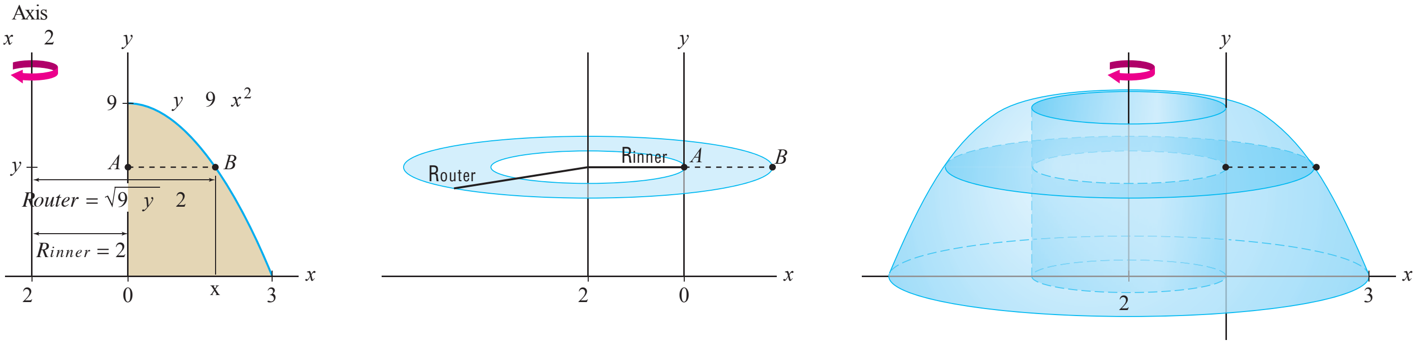


FIGURE 9

The region extends from $y = 0$ to $y = 9$ along the y-axis, so

$$\pi \int_0^9 (R_{\text{outer}}^2 - R_{\text{inner}}^2)\, dy = \pi \int_0^9 \left((\sqrt{9-y}+2)^2 - 2^2\right) dy$$

$$= \pi \int_0^9 \left(9 - y + 4\sqrt{9-y}\right) dy$$

$$= \pi \left(9y - \frac{1}{2}y^2 - \frac{8}{3}(9-y)^{3/2}\right)\Bigg|_0^9 = \frac{225}{2}\pi$$

6.3 SUMMARY

- *Disk method:* When the region between the graph of $f(x)$ and the x-axis for $a \le x \le b$is rotated about the x-axis, we obtain a solid whose vertical cross section is a circle of radius $R = f(x)$ and area $\pi R^2 = \pi f(x)^2$. The volume V of the solid is

$$V = \pi \int_a^b R^2\, dx = \pi \int_a^b f(x)^2\, dx$$

Keep in mind that $f(x)^2$ denotes the square $\big(f(x)\big)^2$.

- *Washer method:* Assume that $f(x) \ge g(x) \ge 0$ for $a \le x \le b$. When we rotate the region between the graphs of $f(x)$ and $g(x)$ about the x-axis, we obtain a solid whose vertical cross section is a washer of outer radius $R_{\text{outer}} = f(x)$ and inner radius $R_{\text{inner}} = g(x)$. The volume V of the solid is

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2)\, dx = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx$$

- *Rotation about an arbitrary horizontal line $y = c$:* The formulas apply but the radii must be modified appropriately. If $f(x) \ge g(x) \ge 0$, then $R_{\text{outer}} = |f(x) - c|$ and $R_{\text{inner}} = |g(x) - c|$. It is helpful to draw the graphs of $f(x)$ and $g(x)$ in order to visualize the disks or washers that are generated.

- These formulas also apply when we rotate about a vertical line $x = c$, but we must integrate along the y axis, and the radii R_{outer} and R_{inner} must be expressed as functions of y.

6.3 EXERCISES

Preliminary Questions

1. Which of the following is a solid of revolution?

(a) Sphere **(b)** Pyramid **(c)** Cylinder **(d)** Cube

2. True or false? When a solid is formed by rotating the region under a graph about the x-axis, the cross sections perpendicular to the x-axis are circular disks.

3. True or false? When a solid is formed by rotating the region between two graphs about the x-axis, the cross sections perpendicular to the x-axis are circular disks.

4. Which of the following integrals expresses the volume of the solid obtained by rotating the area between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around the x-axis [assume $f(x) \geq g(x) \geq 0$]?

(a) $\pi \displaystyle\int_a^b \left(f(x) - g(x)\right)^2 dx$

(b) $\pi \displaystyle\int_a^b \left(f(x)^2 - g(x)^2\right) dx$

Exercises

In Exercises 1–4, (a) sketch the solid obtained by revolving the region under the graph of $f(x)$ about the x-axis over the given interval, (b) describe the cross section perpendicular to the x-axis located at x, and (c) calculate the volume of the solid.

1. $f(x) = x + 1$, $[0, 3]$

2. $f(x) = x^2$, $[1, 3]$

3. $f(x) = \sqrt{x + 1}$, $[1, 4]$

4. $f(x) = x^{-1}$, $[1, 2]$

In Exercises 5–12, find the volume of the solid obtained by rotating the region under the graph of the function about the x-axis over the given interval.

5. $f(x) = x^2 - 3x$, $[0, 3]$

6. $f(x) = \dfrac{1}{x^2}$, $[1, 4]$

7. $f(x) = x^{5/3}$, $[1, 8]$

8. $f(x) = 4 - x^2$, $[0, 2]$

9. $f(x) = \dfrac{2}{x + 1}$, $[1, 3]$

10. $f(x) = \sqrt{x^4 + 1}$, $[1, 3]$

11. $f(x) = e^x$, $[0, 1]$

12. $f(x) = \sqrt{\cos x \sin x}$, $[0, \pi/2]$

13. NEW Which of the integrands (i)-(iv) should be used to compute the volume of the solid obtained by rotating region R in Figure 10 about (a) $y = 2$ and (b) $y = -2$?

(i) $\left(f(x)^2 + 2^2\right) - \left(g(x)^2 + 2^2\right)$ (ii) $\left(f(x) + 2\right)^2 - \left(g(x) + 2\right)^2$

(iii) $\left(f(x)^2 - 2^2\right) - \left(g(x)^2 - 2^2\right)$ (iv) $\left(f(x) - 2\right)^2 - \left(g(x) - 2\right)^2$

14. NEW Which of the integrals (i)-(iv) should be used to compute the volume of the solid obtained by rotating region R in Figure 10 about $y = 9$?

(i) $\left(9 + f(x)\right)^2 - \left(9 + g(x)\right)^2$ (ii) $\left(9 + g(x)\right)^2 - \left(9 + f(x)\right)^2$

(iii) $\left(9 - f(x)\right)^2 - \left(9 - g(x)\right)^2$ (iv) $\left(9 - g(x)\right)^2 - \left(9 - f(x)\right)^2$

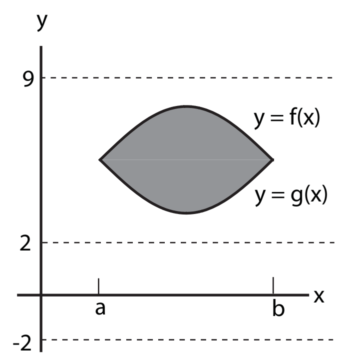


FIGURE 10

In Exercises 15–20, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the x-axis located at x, and (c) find the volume of the solid obtained by rotating the region about the x-axis.

15. $y = x^2 + 2$, $y = 10 - x^2$

16. $y = x^2$, $y = 2x + 3$

17. $y = 16 - x$, $y = 3x + 12$, $x = -1$

18. $y = \dfrac{1}{x}$, $y = \dfrac{5}{2} - x$

19. $y = \sec x$, $y = 0$, $x = -\dfrac{\pi}{4}$, $x = \dfrac{\pi}{4}$

20. $y = \sec x$, $y = \csc x$, $y = 0$, $x = 0$, and $x = \dfrac{\pi}{2}$.

In Exercises 21–24, find the volume of the solid obtained by rotating the region enclosed by the graphs about the y-axis over the given interval.

21. $x = \sqrt{y}$, $x = 0$; $1 \le y \le 4$

22. $x = \sqrt{\sin y}$, $x = 0$; $0 \le y \le \pi$

23. $x = y^2$, $x = \sqrt{y}$; $0 \le y \le 1$

24. $x = 4 - y$, $x = 16 - y^2$; $-3 \le y \le 4$

In Exercises 25–30, find the volume of the solid obtained by rotating region A in Figure 11 about the given axis.

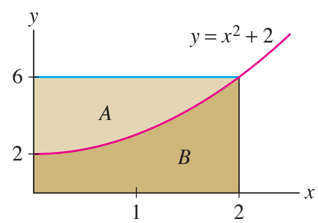


FIGURE 11

25. x-axis

26. $y = -2$

27. $y = 2$

28. y-axis

29. $x = -3$

30. $x = 2$

In Exercises 31–36, find the volume of the solid obtained by rotating region B in Figure 11 about the given axis.

31. x-axis

32. $y = -2$

33. $y = 6$

34. y-axis

Hint for Exercise 34: Express the volume as a sum of two integrals along the y-axis, or use Exercise 28.

35. $x = 2$

36. $x = -3$

In Exercises 37–50, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

37. $y = x^2$, $y = 12 - x$, $x = 0$, about $y = -2$

38. $y = x^2$, $y = 12 - x$, $x = 0$, about $y = 15$

39. $y = 16 - x$, $y = 3x + 12$, $x = 0$, about y-axis

40. $y = 16 - x$, $y = 3x + 12$, $x = 0$, about $x = 2$

41. $y = \dfrac{9}{x^2}$, $y = 10 - x^2$, about x-axis

42. $y = \dfrac{9}{x^2}$, $y = 10 - x^2$, about $y = 12$

43. $y = \dfrac{1}{x}$, $y = \dfrac{5}{2} - x$, about y-axis

44. $x = 2$, $x = 3$, $y = 16 - x^4$, $y = 0$, about y-axis

45. $y = x^3$, $y = x^{1/3}$, about y-axis

46. $y = x^3$, $y = x^{1/3}$, about $x = -2$

47. $y = e^{-x}$, $y = 1 - e^{-x}$, $x = 0$, about $y = 4$

48. $y = \cosh x$, $x = \pm 2$, about x-axis

49. $y^2 = 4x$, $y = x$, $y = 0$, about x-axis

50. $y^2 = 4x$, $y = x$, about $y = 8$

51. GU Sketch the hypocycloid $x^{2/3} + y^{2/3} = 1$ and find the volume of the solid obtained by revolving it about the x-axis.

52. NEW The bowl in Figure 12 (A) is 21 cm high, obtained by rotating the curve (B) as indicated. Estimate the volume capacity of the bowl shown by taking the average of right and left endpoint approximations to the integral with $N = 7$.

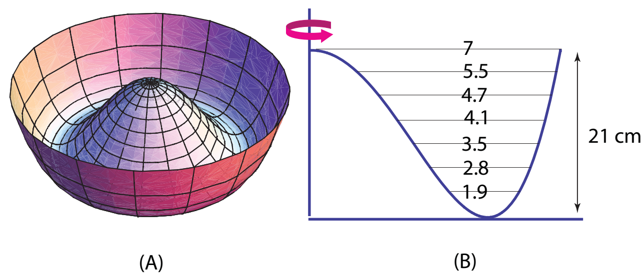


FIGURE 12

53. NEW The region between the graphs of $f(x)$ and $g(x)$ (where $f(x) \ge g(x) \ge 0$) is revolved about the line $y = -3$. Use the midpoint approximation to the integral with values from the following table to estimate the volume V of the resulting solid.

x	0.1	0.3	0.5	0.7	0.9
$f(x)$	8	7	6	7	8
$g(x)$	2	3.5	4	3.5	2

54. The solid generated by rotating the region between the branches of the hyperbola $y^2 - x^2 = 1$ about the x-axis is called a **hyperboloid** (Figure 13). Find the volume of the hyperboloid for $-a \le x \le a$.

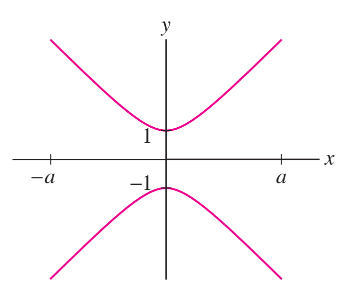


FIGURE 13 The hyperbola with equation $y^2 - x^2 = 1$.

55. A "bead" is formed by removing a cylinder of radius r from the center of a sphere of radius R (Figure 14). Find the volume of the bead with $r = 1$ and $R = 2$.

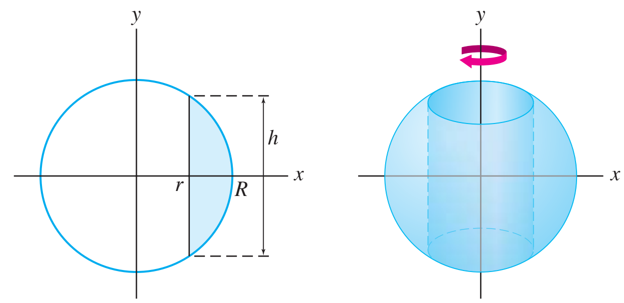


FIGURE 14 A bead is a sphere with a cylinder removed.

Further Insights and Challenges

56. Find the volume V of the bead (Figure 14) in terms of r and R. Then show that $V = \frac{\pi}{6}h^3$, where h is the height of the bead. This formula has a surprising consequence: Since V can be expressed in terms of h alone, it follows that two beads of height 2 in., one formed from a sphere the size of an orange and the other the size of the earth would have the same volume! Can you explain intuitively how this is possible?

57. The solid generated by rotating the region inside the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ around the x-axis is called an **ellipsoid**. Show that the ellipsoid has volume $\frac{4}{3}\pi ab^2$. What is the volume if the ellipse is rotated around the y-axis?

58. A doughnut-shaped solid is called a **torus** (Figure 15). Use the washer method to calculate the volume of the torus obtained by rotating the region inside the circle with equation $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). *Hint:* Evaluate the integral by interpreting it as the area of a circle.

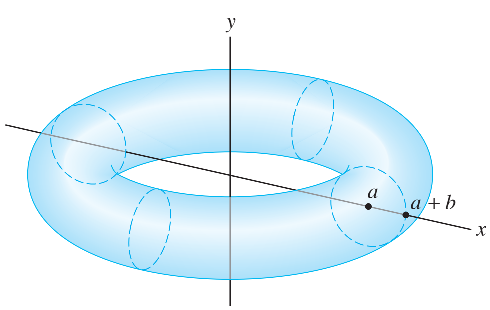


FIGURE 15 Torus obtained by rotating a circle about the y-axis.

59. The curve $y = f(x)$ in Figure 16, called a **tractrix**, has the following property: the tangent line at each point (x, y) on the curve has slope

$$\frac{dy}{dx} = \frac{-y}{\sqrt{1 - y^2}}.$$

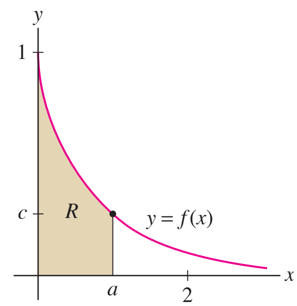


FIGURE 16 The tractrix.

Let R be the shaded region under the graph of $0 \le x \le a$ in Figure 16. Compute the volume V of the solid obtained by revolving R around the x-axis in terms of the constant $c = f(a)$. *Hint:* Use the disk method and the substitution $u = f(x)$ to show that

$$V = \pi \int_c^1 u\sqrt{1 - u^2}\, du$$

60. Verify the formula

$$\int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \frac{1}{6}(x_1 - x_2)^3 \qquad \boxed{3}$$

Then prove that the solid obtained by rotating the shaded region in Figure 17 about the x-axis has volume $V = \frac{\pi}{6}BH^2$, with B and H as in the figure. *Hint:* Let x_1 and x_2 be the roots of $f(x) = ax + b - (mx + c)^2$, where $x_1 < x_2$. Show that

$$V = \pi \int_{x_1}^{x_2} f(x)\, dx$$

and use Eq. (3).

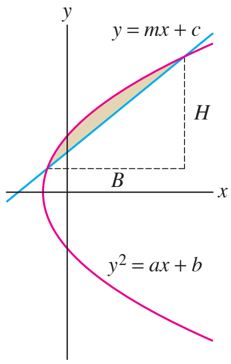


FIGURE 17 The line $y = mx + c$ intersects the parabola $y^2 = ax + b$ at two points above the x-axis.

61. Let R be the region in the unit circle lying above the cut with the line $y = mx + b$ (Figure 18). Assume the points where the line intersects the circle lie above the x-axis. Use the method of Exercise 60 to show that the solid obtained by rotating R about the x-axis has volume $V = \frac{\pi}{6}hd^2$, with h and d as in the figure.

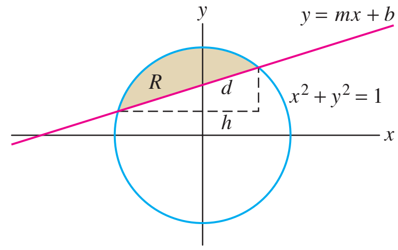


FIGURE 18

6.4 The Method of Cylindrical Shells

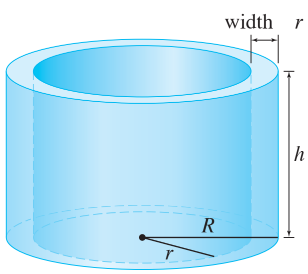


FIGURE 1 The volume of the cylindrical shell is *approximately* $2\pi Rh\,\Delta r$, where $\Delta r = R - r$.

In the previous two sections, we computed volumes by integrating cross-sectional area. The Shell Method is based on a different idea and is more convenient in some cases.

The **Shell Method** uses cylindrical shells like the one in Figure 1 to approximate volumes of revolution. Let us first derive an approximation to the volume of a cylindrical shell of height h, outer radius R, and inner radius r. The shell is obtained by removing a cylinder of radius r from the wider cylinder of radius R, so the shell has volume

$$\pi R^2 h - \pi r^2 h = \pi h(R^2 - r^2) = \pi h(R + r)(R - r) = \pi h(R + r)\Delta r \qquad \boxed{1}$$

where $\Delta r = R - r$ is the shell's width. If the shell is very thin, then R and r are nearly equal and we may replace $(R + r)$ by $2R$ in Eq. (1) to obtain the approximation

$$\text{Volume of shell} \approx 2\pi Rh\,\Delta r = 2\pi(\text{radius}) \times (\text{height of shell}) \times (\text{thickness}) \qquad \boxed{2}$$

Now consider a solid obtained by rotating the region under $y = f(x)$ from $x = a$ to $x = b$ about the y-axis as in Figure 2. The idea is to divide the solid into thin concentric shells. More precisely, we divide $[a, b]$ into N subintervals of length $\Delta x = \frac{b - a}{N}$ with endpoints $x_0, x_1, \dots, x_N$. When we rotate the thin strip of area above $[x_{i-1}, x_i]$ about the y-axis, we obtain a thin shell whose volume we denote by V_i. The total volume V of the solid is equal to $V = \sum_{i=1}^{N} V_i$.

The top rim of the ith thin shell in Figure 2 is curved. However, when Δx is small, we may approximate this thin shell by the cylindrical shell (with flat rim) of height $f(x_i)$, and use (2) to obtain

$$V_i \approx 2\pi x_i f(x_i)\,\Delta x = (\text{circumference})(\text{height of shell})(\text{thickness})$$

$$V = \sum_{i=1}^{N} V_i \approx 2\pi \sum_{i=1}^{N} x_i f(x_i)\,\Delta x$$

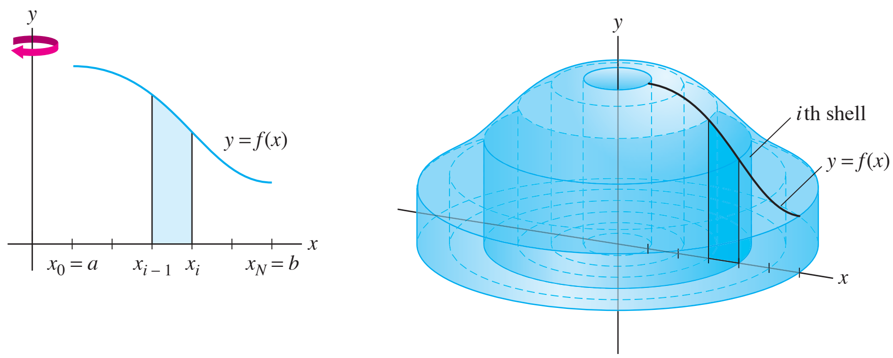


FIGURE 2 The shaded strip, when rotated about the y-axis, generates a "thin shell."

The sum on the right is the volume of the cylindrical approximation to V illustrated in Figure 3. We complete the argument in the usual way. As $N \to \infty$, the accuracy of the approximation improves, and the sum on the right is a right-endpoint approximation that converges to $V = 2\pi \int_a^b x f(x)\, dx$.

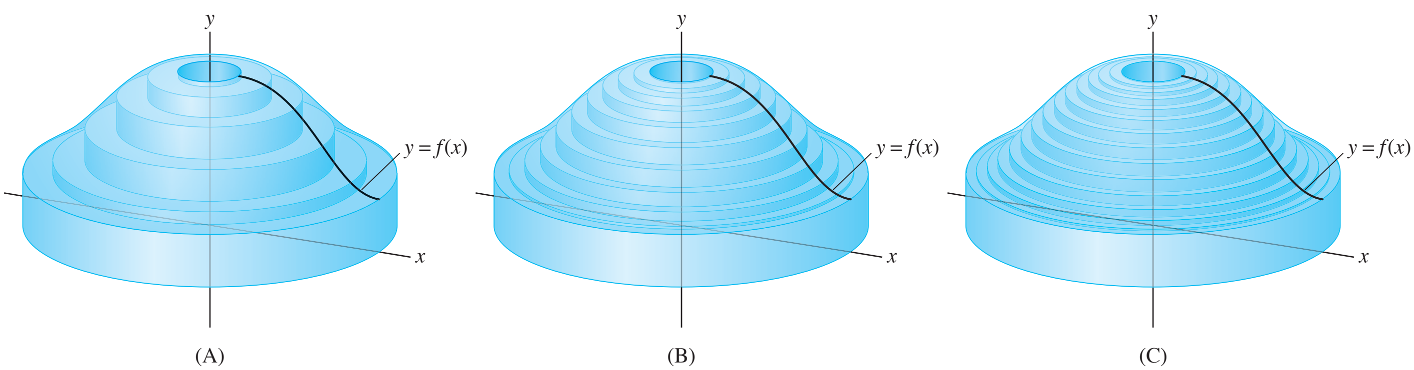


FIGURE 3 The volume is approximated by the sum of volumes of thin cylinders.

In the Shell Method, we integrate with respect to x even though we are rotating about the y-axis.

Volume of a Solid of Revolution: The Shell Method The volume V of the solid obtained by rotating the region under the graph of $y = f(x)$ over the interval $[a, b]$ about the y-axis is equal to

$$V = 2\pi \int_a^b x f(x)\, dx = 2\pi \int_a^b (\text{radius})(\text{height of shell})\, dx \qquad \boxed{3}$$

It would be hard to use the disk method in Example 1. Since the axis of revolution is the y-axis, we would have to integrate with respect to y. This would require finding the inverse function $g(y) = f^{-1}(y)$.

EXAMPLE 1 Find the volume V of the solid obtained by rotating the area under the graph of $f(x) = 1 - 2x + 3x^2 - 2x^3$ over $[0, 1]$ about the y-axis.

Solution The solid is shown in Figure 4. By Eq. (3),

$$V = 2\pi \int_0^1 x f(x)\, dx = 2\pi \int_0^1 x(1 - 2x + 3x^2 - 2x^3)\, dx$$

$$= 2\pi \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5 \right) \Big|_0^1 = \frac{11}{30}\pi$$ ■

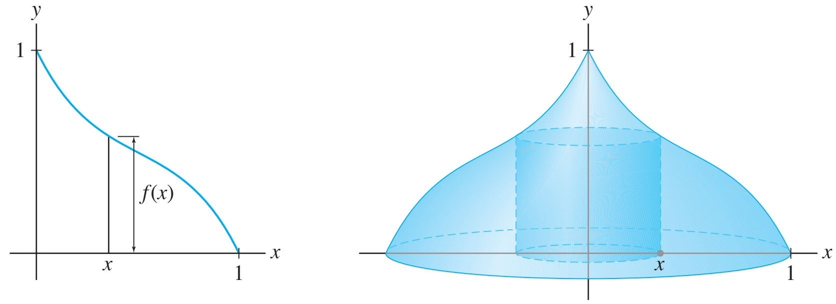


FIGURE 4 The graph of $f(x) = 1 - 2x + 3x^2 - 2x^3$ rotated about the y-axis.

For some solids, it is necessary to modify Eq. (3) in various ways. As a first example, let us rotate the region between the graphs of two functions $f(x)$ and $g(x)$ over $[a, b]$ about the y-axis. Assuming that $f(x) \geq g(x)$, the vertical segment at location x generates a cylindrical shell of radius x and height $f(x) - g(x)$ (Figure 5), so the volume is

$$V = 2\pi \int_a^b \big(\text{radius}\big)\big(\text{height of shell}\big)\, dx = 2\pi \int_a^b x\big(f(x) - g(x)\big)\, dx \qquad \boxed{4}$$

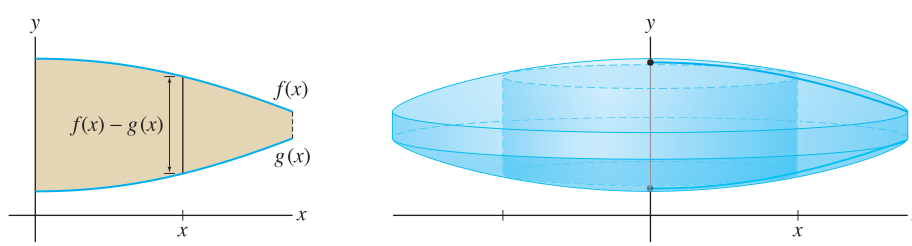


FIGURE 5 The vertical segment at location x generates a cylinder shell of radius x and height $f(x) - g(x)$.

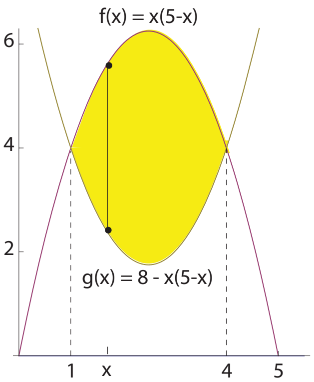


FIGURE 6

EXAMPLE 2 Rotating the Area Between Two Curves Find the volume V of the solid obtained by rotating the area enclosed by the graphs of $f(x) = x(5-x)$ and $g(x) = 8 - x(5-x)$ about the y-axis.

Solution First, find the points of intersection by solving $x(5-x) = 8 - x(5-x)$. We obtain $x(5-x) = 4$ or $x^2 - 5x + 4 = (x-1)(x-4) = 0$, so the curves intersect at $x = 1, 4$. Sketching the graphs as in Figure 6, we see that $f(x) \ge g(x)$ on the interval $[1, 4]$ and

$$\text{height of shell} = f(x) - g(x) = x(5-x) - \big(8 - x(5-x)\big) = 10x - 2x^2 - 8$$

$$V = 2\pi \int_1^4 (\text{radius})(\text{height of shell})\,dx = 2\pi \int_1^4 x\big(10x - 2x^2 - 8\big)\,dx$$

$$= 2\pi \left(\frac{10}{3}x^3 - \frac{1}{2}x^4 - 4x^2\right)\bigg|_1^4 = 2\pi\left(\frac{64}{3} - \left(-\frac{7}{6}\right)\right) = 45\pi$$

EXAMPLE 3 Rotating About a Vertical Axis Use the Shell Method to calculate the volume V of the solid obtained by rotating the region under the graph of $f(x) = x^{-1/2}$ over $[1, 4]$ about the axis $x = -3$.

The reasoning in Example 3 shows that if we rotate the region under $y = f(x)$ over $[a, b]$ about the vertical line $x = c$, then the volume is

$$V = 2\pi \int_a^b (x - c) f(x)\,dx \quad \text{if } c < a$$

$$V = 2\pi \int_a^b (c - x) f(x)\,dx \quad \text{if } c > b$$

Solution

***Step 1.* Warmup.**
If we revolved about the y-axis (that is, $x = 0$), the volume would be

$$V\ (\text{about } y\text{-axis}) = 2\pi \int_1^4 (\text{radius})(\text{height of shell})\,dx = 2\pi \int_1^4 x \cdot x^{-1/2}\,dx$$

***Step 2.* Revolving about $x = -3$.**
The formula is similar, but as we see in Figure 7, the radius of the shell is now $x - (-3) = x + 3$. The height of the shell is still $f(x) = x^{-1/2}$, so

$$V\ (\text{about } x = -3) = 2\pi \int_1^4 (\text{radius})(\text{height of shell})\,dx$$

$$= 2\pi \int_1^4 (x+3)x^{-1/2}\,dx = 2\pi\left(\frac{2}{3}x^{3/2} + 6x^{1/2}\right)\bigg|_1^4 = \frac{64\pi}{3}$$

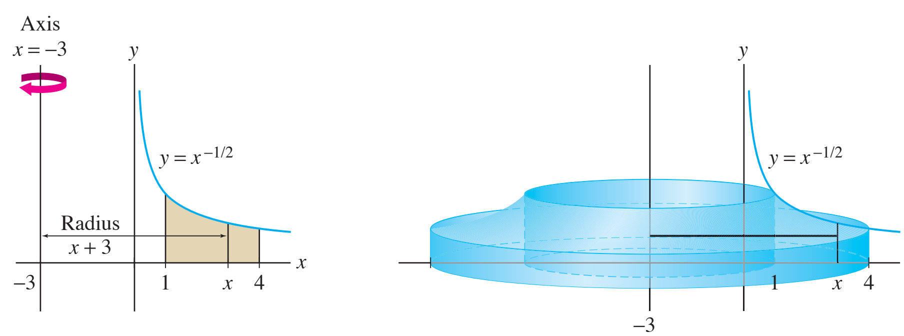


FIGURE 7 Region under the graph of $y = x^{-1/2}$ over $[1, 4]$ rotated about the axis $x = -3$.

The method of cylindrical shells can be applied to rotations about horizontal axes, but in this case, the graph must be described in the form $x = g(y)$.

EXAMPLE 4 Rotating About the x-Axis Use the Shell Method to compute the volume V of the solid obtained by rotating the area under $y = 9 - x^2$ over $[0, 3]$ about the x-axis.

Solution Since we are rotating about the x-axis rather than the y-axis, the Shell Method gives us an integral with respect to y. Therefore, we solve $y = 9 - x^2$ to obtain $x^2 = 9 - y$ or $x = \sqrt{9 - y}$.

The cylindrical shells are generated by *horizontal* segments. The segment $\overline{AB}$ in Figure 8 generates a cylindrical shell of radius y and height $\sqrt{9-y}$ (we still use the term "height" although the shell is horizontal). Using the substitution $u = 9 - y$, $du = -dy$ in the resulting integral, we obtain

REMINDER After making the substitution $u = 9 - y$, the limits of integration must be changed. Since $u(0) = 9$ and $u(9) = 0$, we change $\int_0^9$ to $\int_9^0$.

$$V = 2\pi \int_0^9 (\text{radius})(\text{height of shell})\, dy = 2\pi \int_0^9 y\sqrt{9-y}\, dy = -2\pi \int_9^0 (9-u)\sqrt{u}\, du$$

$$= 2\pi \int_0^9 (9u^{1/2} - u^{3/2})\, du = 2\pi \left(6u^{3/2} - \frac{2}{5}u^{5/2}\right)\Bigg|_0^9 = \frac{648}{5}\pi$$

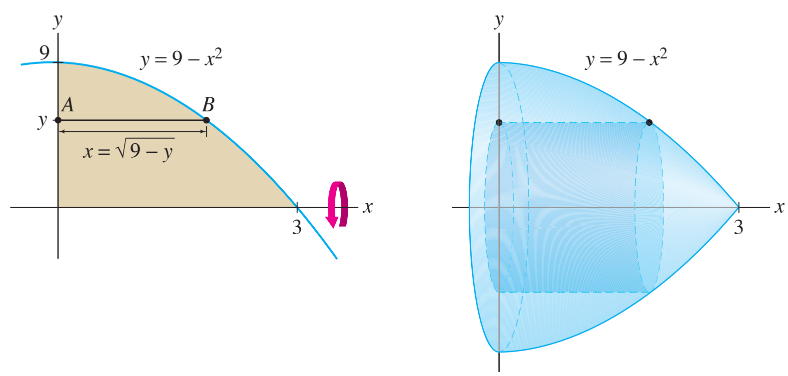


FIGURE 8 Shell generated by a horizontal segment in the region under the graph of $y = 9 - x^2$.

6.4 SUMMARY

- If $f(x) \geq 0$, then the volume V of the solid obtained by rotating the region underneath the graph of $y = f(x)$ over $[a, b]$ about the y-axis is

$$V = 2\pi \int_a^b (\text{radius})(\text{height of shell})\, dx = \boxed{2\pi \int_a^b x f(x)\, dx}$$

- If we revolve the region about the vertical axis $x = c$ rather than the y-axis, then the radius of the shell (distance to the axis of rotation) is no longer x. For example, if $c < a$, the radius is $(x - c)$ and

$$V = 2\pi \int_a^b (\text{radius})(\text{height of shell})\, dx = \begin{cases} 2\pi \int_a^b (x-c) f(x)\, dx & \text{if } c < a \\ 2\pi \int_a^b (c-x) f(x)\, dx & \text{if } c > b \end{cases}$$

- We can use the Shell Method to compute volumes of revolution about the x-axis. It is necessary to express the curve in the form $x = g(y)$:

$$V = 2\pi \int_c^d (\text{radius})(\text{height of shell})\, dy = \boxed{2\pi \int_c^d y\, g(y)\, dy}$$

6.4 EXERCISES

Preliminary Questions

1. Consider the region $\mathcal{R}$ under the graph of the constant function $f(x) = h$ over the interval $[0, r]$. What are the height and radius of the cylinder generated when $\mathcal{R}$ is rotated about:

(a) the x-axis **(b)** the y-axis

2. Let V be the volume of a solid of revolution about the y-axis.

(a) Does the Shell Method for computing V lead to an integral with respect to x or y?

(b) Does the Disk or Washer Method for computing V lead to an integral with respect to x or y?

Exercises

In Exercises 1–10, sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y-axis and find its volume.

1. $f(x) = x^3,\quad [0,1]$

2. $f(x) = \sqrt{x},\quad [0,4]$

3. $f(x) = 3x + 2,\quad [2,4]$

4. $f(x) = 1 + x^2,\quad [1,3]$

5. $f(x) = 4 - x^2,\quad [0,2]$

6. $f(x) = \sqrt{x^2+9},\quad [0,3]$

7. $f(x) = \sin(x^2),\quad [0, \sqrt{\pi}]$

8. $f(x) = x^{-1},\quad [1,3]$

9. $f(x) = x + 1 - 2x^2,\quad [0,1]$

10. $f(x) = \dfrac{x}{\sqrt{1+x^3}},\quad [1,4]$

In Exercises 11–14, use the Shell Method to compute the volume of the solids obtained by rotating the region enclosed by the graphs of the functions about the y-axis.

11. $y = x^2,\quad y = 8 - x^2,\quad x = 0$

12. $y = 8 - x^3,\quad y = 8 - 4x$

13. $y = \sqrt{x},\quad y = x^2$

14. $y = 1 - |x - 1|,\quad y = 0$

GU *In Exercises 15–16, use the Shell Method to compute the volume of rotation of the region enclosed by the curves about the y-axis. Use a computer algebra system or graphing utility to find the points of intersection numerically.*

15. $y = \frac{1}{2}x^2,\quad y = \sin(x^2)$

16. $y = e^{-x^2/2},\quad y = x,\quad x = 0$

In Exercises 17–22, sketch the solid obtained by rotating the region underneath the graph of the function over the interval about the given axis and calculate its volume using the Shell Method.

17. $f(x) = x^3,\quad [0,1],\quad x = 2$

18. $f(x) = x^3,\quad [0,1],\quad x = -2$

19. $f(x) = x^{-4},\quad [-3,-1],\quad x = 4$

20. $f(x) = \dfrac{1}{\sqrt{x^2+1}},\quad [0,2],\quad x = 0$

21. $f(x) = a - bx,\quad [0, a/b],\quad x = -1,\quad a, b > 0$

22. $f(x) = 1 - x^2,\quad [-1,1],\quad x = c$ (with $c > 1$)

In Exercises 23–28, use the Shell Method to calculate the volume of rotation about the x-axis for the region underneath the graph.

23. $y = x,\quad 0 \le x \le 1$

24. $y = 4 - x^2,\quad 0 \le x \le 2$

25. $y = x^{1/3} - 2,\quad 8 \le x \le 27$

26. $y = x^{-1},\quad 1 \le x \le 4$. Sketch the region and express the volume as a sum of two integrals.

27. $y = x^{-2},\quad 2 \le x \le 4$

28. $y = \sqrt{x},\quad 1 \le x \le 4$

29. Use both the Shell and Disk Methods to calculate the volume of the solid obtained by rotating the region under the graph of $f(x) = 8 - x^3$ for $0 \le x \le 2$ about:

(a) the x-axis **(b)** the y-axis

30. Sketch the solid of rotation about the y-axis for the region under the graph of the constant function $f(x) = c$ (where $c > 0$) for $0 \le x \le r$.

(a) Find the volume without using integration.

(b) Use the Shell Method to compute the volume.

31. Assume that the graph in Figure 9(A) can be described by both $y = f(x)$ and $x = h(y)$. Let V be the volume of the solid obtained by rotating the region under the curve about the y-axis.

(a) Describe the figures generated by rotating segments $\overline{AB}$ and $\overline{CB}$ about the y-axis.

(b) Set up integrals that compute V by the Shell and Disk Methods.

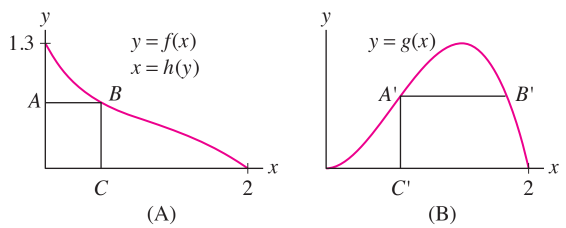


FIGURE 9

32. Let W be the volume of the solid obtained by rotating the region under the curve in Figure 9(B) about the y-axis.

(a) Describe the figures generated by rotating segments $\overline{A'B'}$ and $\overline{A'C'}$ about the y-axis.

(b) Set up an integral that computes W by the Shell Method.

(c) Explain the difficulty in computing W by the Washer Method.

In Exercises 33–38, use the Shell Method to find the volume of the solid obtained by rotating region A in Figure 10 about the given axis.

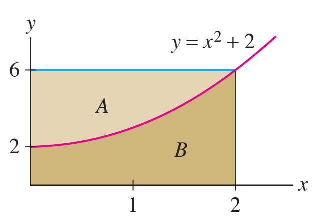


FIGURE 10

33. y-axis

34. $x = -3$

35. $x = 2$

36. x-axis

37. $y = -2$

38. $y = 6$

In Exercises 39–44, use the Shell Method to find the volumes of the solids obtained by rotating region B in Figure 10 about the given axis.

39. y-axis

40. $x = -3$

41. $x = 2$

42. x-axis

43. $y = -2$

44. $y = 8$

45. Use the Shell Method to compute the volume of a sphere of radius r.

46. Use the Shell Method to calculate the volume V of the "bead" formed by removing a cylinder of radius r from the center of a sphere of radius R (compare with Exercise 55 in Section 6.3).

47. Use the Shell Method to compute the volume of the torus obtained by rotating the interior of the circle $(x - a)^2 + y^2 = r^2$ about the y-axis, where $a > r$. *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

48. Use the Shell or Disk Method (whichever is easier) to compute the volume of the solid obtained by rotating the region in Figure 11 about:

(a) the x-axis

(b) the y-axis

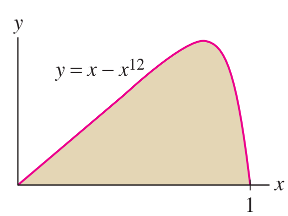


FIGURE 11

49. Use the most convenient method to compute the volume of the solid obtained by rotating the region in Figure 12 about the axis:

(a) $x = 4$

(b) $y = -2$

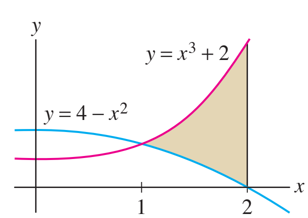


FIGURE 12

Further Insights and Challenges

50. The surface area of a sphere of radius r is $4\pi r^2$. Use this to derive the formula for the volume V of a sphere of radius R in a new way.

(a) Show that the volume of a thin spherical shell of inner radius r and thickness Δx is approximately $4\pi r^2 \Delta x$.

(b) Approximate V by decomposing the sphere of radius R into N thin spherical shells of thickness $\Delta x = R/N$.

(c) Show that the approximation is a Riemann sum which converges to an integral. Evaluate the integral.

51. Let R be the region bounded by the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ (Figure 13). Show that the solid obtained by rotating R about the y-axis (called an **ellipsoid**) has volume $\frac{4}{3}\pi a^2 b$.

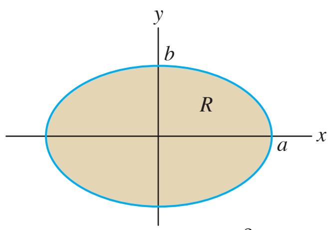


FIGURE 13 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

52. The bell-shaped curve in Figure 14 is the graph of a certain function $y = f(x)$ with the following property: The tangent line at a point (x, y) on the graph has slope $dy/dx = -xy$. Let R be the shaded region under the graph for $0 \le x \le a$ in Figure 14. Use the Shell Method and the substitution $u = f(x)$ to show that the solid obtained by revolving R about the y-axis has volume $V = 2\pi(1 - c)$, where $c = f(a)$. Observe that as $c \to 0$, the region R becomes infinite but the volume V approaches 2π.

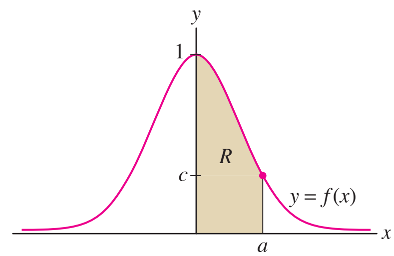


FIGURE 14 The bell-shaped curve.

6.5 Work and Energy

"For those who want some proof that physicists are human, the proof is in the idiocy of all the different units which they use for measuring energy."

—Richard Feynman,
The Character of Physical Law

All physical tasks, from boiling water to turning on a cell phone, require an expenditure of energy. When a force is applied to an object to move it, the energy expended is called **work**. If a *constant* force F is applied through a distance d, then the work W is defined as "force times distance" (Figure 1)

$$W = F \cdot d \qquad (1)$$

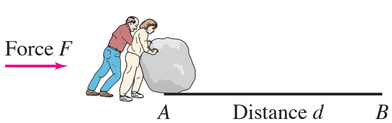


FIGURE 1 The work expended to move the object from A to B is $W = F \cdot d$.

In the metric system, the unit of force is the *newton* (abbreviated N), defined as 1 kg-m/s^2. Energy and work are both measured in units of the *joule* (J), equal to 1 N-m. In the British system, the unit of force is the pound, and both energy and work are measured in foot-pounds (ft-lb). Another unit of energy is the *calorie*. One ft-lb is approximately 0.738 J or 3.088 calories.

To become familiar with the units, let's calculate the work W required to lift a 2-kg stone 3 m above the ground. Gravity pulls down on the stone of mass m with a force equal to $-mg$, where $g = 9.8$ m/s^2. Therefore, lifting the stone requires an upward vertical force $F = mg$, and the work expended is

$$W = \underbrace{(mg)h}_{F \cdot d} = (2\text{ kg})(9.8\text{ m/s}^2)(3\text{ m}) = 58.8\text{ J}$$

While the kilogram is a unit of mass, the pound is a unit of force rather than mass, so the factor g does not appear when computing work against gravity in the British system. The work required to lift a 2-lb stone 3 ft above ground is

$$W = \underbrace{(2 \text{ lb})(3 \text{ ft})}_{F \cdot d} = 6 \text{ ft-lb}$$

We use integration to calculate work when the force is not constant. Suppose that the force $F(x)$ varies as the object moves from a to b along the x-axis. Then Eq. (1) does not apply directly, but we may break up the task into a large number of smaller tasks where Eq. (1) gives a good approximation. Divide $[a, b]$ into N subintervals of length $\Delta x = \dfrac{b-a}{N}$, with endpoints: $a = x_0, x_1, x_2, \dots, x_{N-1}, x_N = b$. Let W_i be the work required to move the object from x_{i-1} to x_i (Figure 2). If Δx is small, then the force $F(x)$ is nearly constant on the interval $[x_{i-1}, x_i]$ with value $F(x_i)$, so $W_i \approx F(x_i)\,\Delta x$. Summing the contributions, we obtain

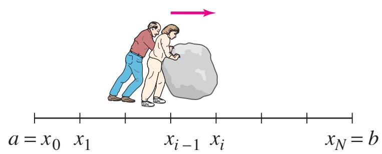


FIGURE 2 The work to move an object from x_{i-1} to x_i is approximately $F(x_i)\,\Delta x$.

$$W = \sum_{i=1}^{N} W_i \approx \underbrace{\sum_{i=1}^{N} F(x_i)\,\Delta x}_{\text{Right-endpoint approximation}}$$

The sum on the right is a right-endpoint approximation converging to $\int_a^b F(x)\,dx$. This leads to the following definition.

DEFINITION Work The work performed in moving an object along the x-axis from a to b by applying a force of magnitude $F(x)$ is

$$W = \int_a^b F(x)\,dx \qquad \boxed{2}$$

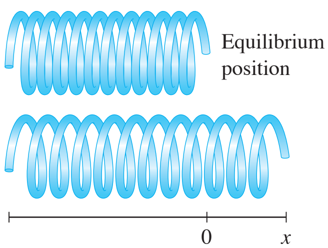


FIGURE 3 According to Hooke's Law, a spring stretched x units past equilibrium exerts a restoring force $-kx$ in the opposite direction.

One typical calculation involves finding the work required to stretch a spring. Assume that the end of the spring has position $x = 0$ at equilibrium, when no force is acting (Figure 3). The spring may be stretched x units (or compressed if $x < 0$). **Hooke's Law** states that the spring exerts a restoring force $-kx$ in the opposite direction, where k is the **spring constant**, measured in units of kilograms per second squared.

Hooke's Law is named after the English scientist, inventor, and architect Robert Hooke (1635–1703) who made important discoveries in physics, astronomy, chemistry, and biology. He was a pioneer in the use of the microscope to study organisms. Unfortunately, Hooke was involved in several bitter disputes with other scientists, most notably with his contemporary Isaac Newton. Newton was furious when Hooke criticized his work on optics. Later, Hooke told Newton that he believed Kepler's Laws would follow from an inverse square law of gravitation. Newton refused to acknowledge Hooke's contributions in his masterwork Principia. *It was in a letter to Hooke that Newton made his famous remark "If I have seen further it is by standing on the shoulders of giants."*

EXAMPLE 1 Hooke's Law Assuming a spring constant of $k = 400$ kg/s^2, find the work (in joules) required to **(a)** stretch the spring 10 cm beyond equilibrium and **(b)** compress the spring 2 additional cm when it is already compressed 3 cm.

Solution By Hooke's Law, the spring exerts a restoring force of $-400x$ N when it is stretched x units. Therefore, we must apply a force $F(x) = 400x$ N to stretch the spring further. To compute the work in joules, we must convert from centimeters to meters since 1 J is equal to a Newton-meter. Therefore,

(a) The work required to stretch the spring 10 cm (0.1 m) beyond equilibrium is

$$W = \int_0^{0.1} 400x\,dx = 200x^2\Big|_0^{0.1} = 2 \text{ J}$$

(b) If the spring is at position $x = -0.03$ m, then the work W required to compress it further to $x = -0.05$ m is

$$W = \int_{-0.03}^{-0.05} 400x\,dx = 200x^2\Big|_{-0.03}^{-0.05} = 0.5 - 0.18 = 0.32 \text{ J}$$

Note that we integrate from right to left (the lower limit -0.03 is *larger* than the upper limit -0.05) because we're compressing the spring to the left. ■

In the next two examples, we compute work in a different way. In these examples, the formula $W = \int_a^b F(x)\,dx$ cannot be used because we are not moving a single object through a fixed distance. Rather, each thin layer of the object is moved through a different distance. We compute total work by "summing" (i.e., *integrating*) the work performed on each thin layer.

On the earth's surface, work against gravity is equal to the force mg times the vertical distance through which the object is moved. No work against gravity is done when an object is moved sideways.

■ **EXAMPLE 2** **Building a Cement Column** Compute the work (against gravity) required to build a cement column of height 5 m and square base of side 2 m. Assume that cement has density 1,500 kg/m^3.

Solution Think of the column as a stack of N thin layers of width $\Delta y = 5/N$. The work consists of lifting up these layers and placing them on the stack (Figure 4), but the work performed on a given layer depends on how high we lift it. First, let us compute the gravitational force on a thin layer of width Δy:

$$\begin{aligned}
&\text{Volume of layer} = \text{area} \times \text{width} &&= 4\,\Delta y \text{ m}^3\\
&\text{Mass of layer} \quad = \text{density} \times \text{volume} &&= 1{,}500 \cdot 4\,\Delta y \text{ kg}\\
&\text{Force on layer} \quad = g \times \text{mass} &&= 9.8 \cdot 1{,}500 \cdot 4\,\Delta y = 58{,}800\,\Delta y \text{ N}
\end{aligned}$$

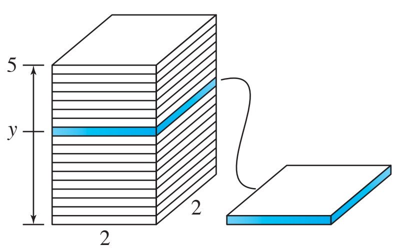


FIGURE 4 Total work is the sum of the work performed on each layer of the column.

The work required to raise this layer to height y is approximately equal to the force times the distance y, that is, $58{,}800y\,\Delta y$. We set $W(y) = 58{,}800y$ and write

Work performed lifting layer to height $y \approx W(y)\,\Delta y$

This is only an approximation because the layer has a nonzero width and the cement particles at the top have been lifted a little bit higher than those at the bottom.

The ith layer is lifted to height $y_i = i\,\Delta y$, and the total work performed is

$$W \approx \sum_{i=1}^{N} W(y_i)\,\Delta y$$

The sum on the right is a right-endpoint approximation to $\int_0^5 W(y)\,dy$. Letting N tend to ∞, we obtain

$$W = \int_0^5 W(y)\,dy = \int_0^5 58{,}800y\,dy = 58{,}800\frac{y^2}{2}\Big|_0^5 = 735{,}000 \text{ J}$$ ■

■ **EXAMPLE 3** **Pumping Water out of a Tank** A spherical tank of radius R meters with a small hole at the top is filled with water. How much work (against gravity) is done pumping the water out through the hole? The density of water is 1,000 kg/m^3.

Solution As in the previous example, we divide the sphere into N thin layers of width $\Delta y = \frac{2R}{N}$. We place the origin of our coordinate system at the center of the sphere because this leads to a simple formula for the radius x of the cross section at height y. Referring to Figure 5, we see that $x = \sqrt{R^2 - y^2}$ by the Pythagorean Theorem.

***Step 1.* Approximate the work performed on a single layer.**
The thin layer located at y is nearly cylindrical of height Δy and radius $x = \sqrt{R^2 - y^2}$, so its volume is

$$\text{Volume of layer at } y \approx \pi x^2 \Delta y = \pi(R^2 - y^2)\,\Delta y \text{ m}^3$$

Furthermore,

$$\text{Mass of } i\text{th layer} = \text{density} \times \text{volume} \approx 1{,}000\pi(R^2 - y^2)\,\Delta y \text{ kg}$$
$$\text{Force on } i\text{th layer} = g \times \text{mass} \approx (9.8)1{,}000\pi(R^2 - y^2)\,\Delta y \text{ N}$$

We must llift up the water in this layer a vertical distance $R - y$ (no work against gravity is required to move an object sideways), so the work performed on the layer is approximately

$$\overbrace{9{,}800\pi(R^2 - y^2)\,\Delta y}^{\text{Force against gravity}} \cdot \overbrace{(R - y)}^{\text{Vertical distance moved}} = \overbrace{9{,}800\pi(R^3 - R^2y - Ry^2 + y^3)}^{\text{Call this } W(y)}\,\Delta y$$

Let y_i be the height of the ith layer. Then with $W(y)$ as indicated,

$$\boxed{\text{Work performed on } i\text{th layer} \approx W(y_i)\,\Delta y}$$

***Step 2.* Integrate the work performed on the layers.**
As in Example 2, the total work W is the sum of the work performed on the N layers. Thus $W \approx \sum_{i=1}^{N} W(y_i)\,\Delta y$ and in the limit as $N \to \infty$, we obtain the integral of $W(y)$. The integral extends from $-R$ to R because the y-coordinate along the sphere varies from $-R$ to R:

$$W = \int_{-R}^{R} W(y)\,dy = 9{,}800\pi \int_{-R}^{R} (R^3 - R^2y - Ry^2 + y^3)\,dy \qquad \boxed{3}$$

$$= 9{,}800\pi \left(R^3y - \frac{1}{2}R^2y^2 - \frac{1}{3}Ry^3 + \frac{1}{4}y^4 \right) \Bigg|_{-R}^{R} = \frac{39{,}200\pi}{3} R^4 \text{ J}$$

6.5 SUMMARY

- The work W performed when a force F is applied to move an object along a straight line:

$$\text{Constant force:} \quad W = F \cdot d, \qquad \text{Variable force:} \quad W = \int_a^b F(x)\,dx$$

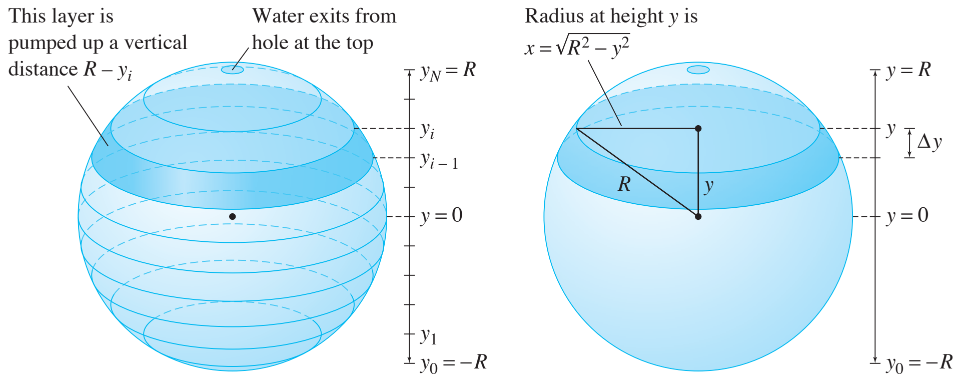


FIGURE 5 The radius of a thin layer at height y is $x = \sqrt{R^2 - y^2}$.

- In some cases, the work is computed by decomposing an object into N thin layers of thickness $\Delta y = \dfrac{b-a}{N}$ (where the object extends from $y = a$ to $y = b$). We approximate the work W_i performed on the ith layer as $W_i \approx W(y_i)\,\Delta y$ for some function $W(y)$. The total work is equal to $W = \int_a^b W(y)\,dy$.

6.5 EXERCISES

Preliminary Questions

1. Why is integration needed to compute the work performed in stretching a spring?

2. Why is integration needed to compute the work performed in pumping water out of a tank but not to compute the work performed in lifting up the tank?

3. Which of the following represents the work required to stretch a spring (with spring constant k) a distance x beyond its equilibrium position: kx, $-kx$, $\frac{1}{2}mk^2$, $\frac{1}{2}kx^2$, or $\frac{1}{2}mx^2$?

Exercises

1. How much work is done raising a 4-kg mass to a height of 16 m above ground?

2. How much work is done raising a 4-lb mass to a height of 16 ft above ground?

In Exercises 3–6, compute the work (in joules) required to stretch or compress a spring as indicated, assuming that the spring constant is $k = 150$ kg/s^2.

3. Stretching from equilibrium to 12 cm past equilibrium

4. Compressing from equilibrium to 4 cm past equilibrium

5. Stretching from 5 to 15 cm past equilibrium

6. Compressing the spring 4 more cm when it is already compressed 5 cm

7. If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to stretch it 15 cm beyond equilibrium?

8. If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to compress it 5 cm beyond equilibrium?

9. If 10 ft-lb of work are needed to stretch a spring 1 ft beyond equilibrium, how far will the spring stretch if a 10-lb weight is attached to its end?

10. Show that the work required to stretch a spring from position a to position b is $\frac{1}{2}k(b^2 - a^2)$, where k is the spring constant. How do you interpret the negative work obtained when $|b| < |a|$?

In Exercises 11–14, calculate the work against gravity required to build the structure out of brick using the method of Examples 2 and 3. Assume that brick has density 80 lb/ft^3.

11. A tower of height 20 ft and square base of side 10 ft

12. A cylindrical tower of height 20 ft and radius 10 ft

13. A 20-ft-high tower in the shape of a right circular cone with base of radius 4 ft

14. A structure in the shape of a hemisphere of radius 4 ft

15. Built around 2600 BCE, the Great Pyramid of Giza in Egypt is 485 ft high (due to erosion, its current height is slightly less) and has a square base of side 755.5 ft (Figure 6). Find the work needed to build the pyramid if the density of the stone is estimated at 125 lb/ft^3.

FIGURE 6 The Great Pyramid in Giza, Egypt.

In Exercises 16–20, calculate the work (in joules) required to pump all of the water out of the tank. Assume that the tank is full, distances are measured in meters, and the density of water is 1,000 kg/m^3.

16. The box in Figure 7; water exits from a small hole at the top.

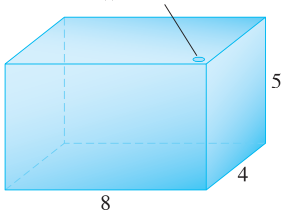


FIGURE 7

17. The hemisphere in Figure 8; water exits from the spout as shown.

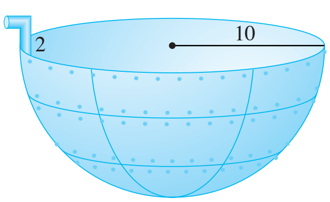


FIGURE 8

18. The conical tank in Figure 9; water exits through the spout as shown.

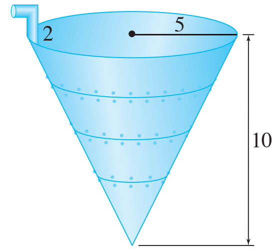


FIGURE 9

19. The horizontal cylinder in Figure 10; water exits from a small hole at the top. *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

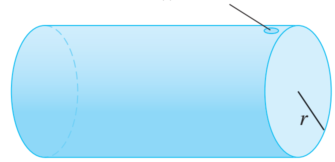


FIGURE 10

20. The trough in Figure 11; water exits by pouring over the sides.

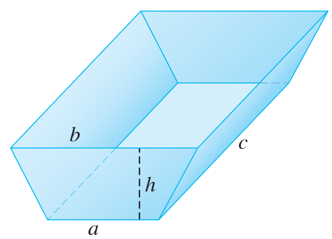


FIGURE 11

21. Find the work W required to empty the tank in Figure 7 if it is half full of water.

22. Assume the tank in Figure 7 is full of water and let W be the work required to pump out half of the water. Do you expect W to equal the work computed in Exercise 21? Explain and then compute W.

23. Find the work required to empty the tank in Figure 9 if it is half full of water.

24. Assume the tank in Figure 9 is full of water and find the work required to pump out half of the water.

25. Assume that the tank in Figure 9 is full.

(a) Calculate the work $F(y)$ required to pump out water until the water level has reached level y.

(b) CAS Plot $F(y)$.

(c) What is the significance of $F'(y)$ as a rate of change?

(d) CAS If your goal is to pump out all of the water, at which water level y_0 will half of the work be done?

26. How much work is done lifting a 25-ft chain over the side of a building (Figure 12)? Assume that the chain has a density of 4 lb/ft. *Hint:* Break up the chain into N segments, estimate the work performed on a segment, and compute the limit as $N \to \infty$ as an integral.

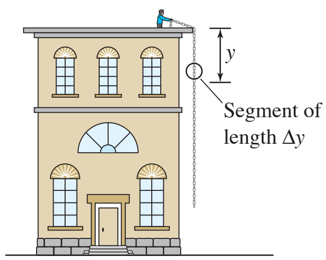


FIGURE 12 The small segment of the chain of length Δy located y feet from the top is lifted through a vertical distance y.

27. How much work is done lifting a 3-m chain over the side of a building if the chain has mass density 4 kg/m?

28. An 8-ft chain weighs 16 lb. Find the work required to lift the chain over the side of a building.

29. A 20-ft chain with mass density 3 lb/ft is initially coiled on the ground. How much work is performed in lifting the chain so that it is fully extended (and one end touches the ground)?

30. How much work is done lifting a 20-ft chain with mass density 3 lb/ft (initially coiled on the ground) so that its top end is 30 ft above the ground?

31. A 1,000-lb wrecking ball hangs from a 30-ft cable of density 10 lb/ft attached to a crane. Calculate the work done if the crane lifts the ball from ground level to 30 ft in the air by drawing in the cable.

In Exercises 32–34, use Newton's Universal Law of Gravity, according to which the gravitational force between two objects of mass m and M separated by a distance r has magnitude GMm/r^2, where $G = 6.67 \times 10^{-11}$ $\mathrm{m^3 kg^{-1} s^{-1}}$. Although the Universal Law refers to point masses, Newton proved that it also holds for uniform spherical objects, where r is the distance between their centers.

32. Two spheres of mass M and m are separated by a distance r_1. Show that the work required to increase the separation to a distance r_2 is equal to $W = GMm(r_1^{-1} - r_2^{-1})$.

33. Use the result of Exercise 32 to calculate the work required to place a 2,000-kg satellite in an orbit 1,200 km above the surface of the earth. Assume that the earth is a sphere of mass $M_e = 5.98 \times 10^{24}$ kg and radius $r_e = 6.37 \times 10^6$ m. Treat the satellite as a point mass.

34. Use the result of Exercise 32 to compute the work required to move a 1,500-kg satellite from an orbit 1,000 to 1,500 km above the surface of the earth.

35. Assume that the pressure P and volume V of the gas in a 30-in. cylinder of radius 3 in. with a movable piston are related by $PV^{1.4} = k$, where k is a constant (Figure 13). When the cylinder is full, the gas pressure is 200 lb/in.^{2}.

(a) Calculate k.

(b) Calculate the force on the piston as a function of the length x of the column of gas (the force is PA, where A is the piston's area).

(c) Calculate the work required to compress the gas column from 30 to 20 in.

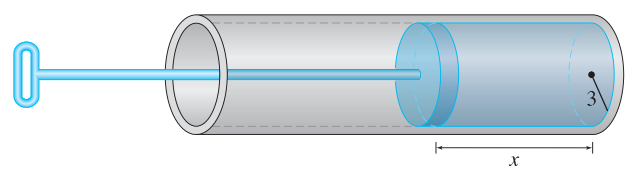


FIGURE 13 Gas in a cylinder with a piston.

Further Insights and Challenges

36. A 20-ft chain with linear mass density

$$\rho(x) = 0.02x(20 - x) \text{ lb/ft}$$

lies on the ground.

(a) How much work is done lifting the chain so that it is fully extended (and one end touches the ground)?

(b) How much work is done lifting the chain so that its top end has a height of 30 ft?

37. Work-Kinetic Energy Theorem The **kinetic energy** of an object of mass m moving with velocity v is KE $= \frac{1}{2}mv^2$.

(a) Suppose that the object moves from x_1 to x_2 during the time interval $[t_1, t_2]$ due to a net force $F(x)$ acting along the interval $[x_1, x_2]$. Let $x(t)$ be the position of the object at time t. Use the Change of Variables formula to show that the work performed is equal to

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{t_1}^{t_2} F(x(t))v(t)\,dt$$

(b) By Newton's Second Law, $F(x(t)) = ma(t)$, where $a(t)$ is the acceleration at time t. Show that

$$\frac{d}{dt}\left(\frac{1}{2}mv(t)^2\right) = F(x(t))v(t)$$

(c) Use the FTC to show that the change in kinetic energy during the time interval $[t_1, t_2]$ is equal to

$$\int_{t_1}^{t_2} F(x(t))v(t)\,dt.$$

(d) Prove the Work-Kinetic Energy Theorem: The change in KE is equal to the work W performed.

38. A model train of mass 0.5 kg is placed at one end of a straight 3-m electric track. Assume that a force $F(x) = 3x - x^2$ N acts on the train at distance x along the track. Use the Work-Kinetic Energy Theorem (Exercise 37) to determine the velocity of the train when it reaches the end of the track.

39. With what initial velocity v_0 must we fire a rocket so it attains a maximum height r above the earth? *Hint:* Use the results of Exercises 32 and 37. As the rocket reaches its maximum height, its KE decreases from $\frac{1}{2}mv_0^2$ to zero.

40. With what initial velocity must we fire a rocket so it attains a maximum height of $r = 20$ km above the surface of the earth?

41. Calculate **escape velocity,** the minimum initial velocity of an object to ensure that it will continue traveling into space and never fall back to earth (assuming that no force is applied after takeoff). *Hint:* Take the limit as $r \to \infty$ in Exercise 39.

CHAPTER REVIEW EXERCISES

In Exercises 1–6, find the area of the region bounded by the graphs of the functions.

1. $y = \sin x, \quad y = \cos x, \quad 0 \le x \le \frac{5\pi}{4}$

2. $f(x) = x^3 - 2x^2 + x, \quad g(x) = x^2 - x$

3. $f(x) = x^2 + 2x, \quad g(x) = x^2 - 1, \quad h(x) = x^2 + x - 2$

4. $f(x) = \sin x, \quad g(x) = \sin 2x, \quad \frac{\pi}{3} \le x \le \pi$

5. $y = e^x, \quad y = 1 - x, \quad x = 1$

6. $y = \cosh 1 - \cosh x, \quad y = \cosh x - \cosh 1$

In Exercises 7–10, sketch the region bounded by the graphs of the functions and find its area.

7. $f(x) = x^3 - x^2 - x + 1, \quad g(x) = \sqrt{1 - x^2}, \quad 0 \le x \le 1$
Hint: Use geometry to evaluate the integral.

8. $x = \frac{1}{2}y, \quad x = y\sqrt{1 - y^2}, \quad 0 \le y \le 1$

9. $y = 4 - x^2, \quad y = 3x, \quad y = 4$

10. $x = y^3 - 2y^2 + y, \quad x = y^2 - y$

11. GU Use a graphing utility to locate the points of intersection of $y = e^{-x}$ and $y = 1 - x^2$ and find the area between the two curves (approximately).

12. Figure 1 shows a solid whose horizontal cross section at height y is a circle of radius $(1 + y)^{-2}$ for $0 \le y \le H$. Find the volume of the solid.

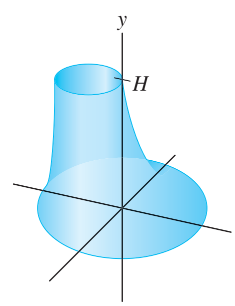


FIGURE 1

13. Find the total weight of a 3-ft metal rod of linear density

$$\rho(x) = 1 + 2x + \frac{2}{9}x^3 \text{ lb/ft}.$$

14. Find the flow rate (in the correct units) through a pipe of diameter 6 cm if the velocity of fluid particles at a distance r from the center of the pipe is $v(r) = (3 - r)$ cm/s.

In Exercises 15–20, find the average value of the function over the interval.

15. $f(x) = x^3 - 2x + 2, \quad [-1, 2]$

16. $f(x) = \sqrt{9 - x^2}, \quad [0, 3]$ *Hint:* Use geometry to evaluate the integral.

17. $f(x) = |x|, \quad [-4, 4]$

18. $f(x) = x[x], \quad [0, 3]$

19. $f(x) = x\cosh(x^2), \quad [0, 1]$

20. $f(x) = \frac{e^x}{1 + e^{2x}}, \quad \left[0, \frac{1}{2}\right]$

21. The average value of $g(t)$ on $[2, 5]$ is 9. Find $\int_2^5 g(t)\,dt$.

22. For all $x \geq 0$, the average value of $R(x)$ over $[0, x]$ is equal to x. Find $R(x)$.

23. Use the Shell Method to find the volume of the solid obtained by revolving the region between $y = x^2$ and $y = mx$ about the x-axis (Figure 2).

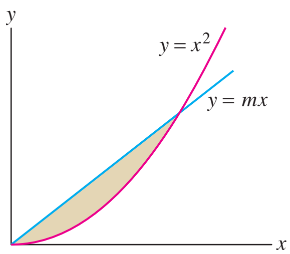


FIGURE 2

24. Use the washer method to find the volume of the solid obtained by revolving the region between $y = x^2$ and $y = mx$ about the y-axis (Figure 2).

25. Let R be the intersection of the circles of radius 1 centered at $(1, 0)$ and $(0, 1)$. Express as an integral (but do not evaluate): **(a)** the area of R and **(b)** the volume of revolution of R about the x-axis.

26. Use the Shell Method to set up an integral (but do not evaluate) expressing the volume of the solid obtained by rotating the region under $y = \cos x$ over $[0, \pi/2]$ about the line $x = \pi$.

In Exercises 27–35, find the volume of the solid obtained by rotating the region enclosed by the curves about the given axis.

27. $y = 2x, \quad y = 0, \quad x = 8; \quad x\text{-axis}$

28. $y = 2x, \quad y = 0, \quad x = 8; \quad \text{axis } x = -3$

29. $y = x^2 - 1, \quad y = 2x - 1, \quad \text{axis } x = -2$

30. $y = x^2 - 1, \quad y = 2x - 1, \quad \text{axis } y = 4$

31. $y^2 = x^3, \quad y = x, \quad x = 8; \quad \text{axis } x = -1$

32. $y^2 = x^{-1}, \quad x = 1, \quad x = 3; \quad \text{axis } y = -3$

33. $y = -x^2 + 4x - 3, \quad y = 0; \quad \text{axis } y = -1$

34. $x = 4y - y^3, \quad y = 0, \quad y = 2; \quad y\text{-axis}$

35. $y^2 = x^{-1}, \quad x = 1, \quad x = 3; \quad \text{axis } x = -3$

In Exercises 36–38, the regions refer to the graph of the hyperbola $y^2 - x^2 = 1$ in Figure 3. Calculate the volume of revolution about both the x- and y-axes.

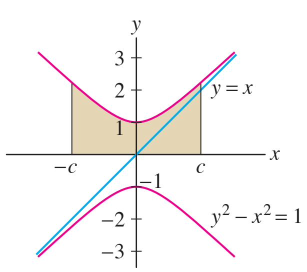


FIGURE 3

36. The shaded region between the upper branch of the hyperbola and the x-axis for $-c \leq x \leq c$.

37. The region between the upper branch of the hyperbola and the line $y = x$ for $0 \leq x \leq c$.

38. The region between the upper branch of the hyperbola and $y = 2$.

39. Let $a > 0$. Show that when the region between $y = a\sqrt{x - ax^2}$ and the x-axis is rotated about the x-axis, the resulting volume is independent of the constant a.

40. A spring whose equilibrium length is 15 cm exerts a force of 50 N when it is stretched to 20 cm. Find the work required to stretch the spring from 22 to 24 cm.

In Exercises 41–42, water is pumped into a spherical tank of radius 5 ft from a source located 2 ft below a hole at the bottom (Figure 4). The density of water is 64.2 lb/ft^3.

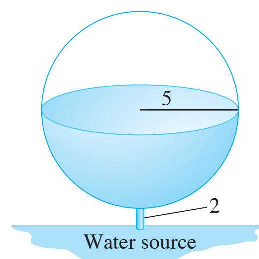


FIGURE 4

41. Calculate the work required to fill the tank.

42. Calculate the work $F(h)$ required to fill the tank to height h ft from the bottom of the sphere.

43. A container weighing 50 lb is filled with 20 ft^3 of water. The container is raised vertically at a constant speed of 2 ft/s for 1 min, during which time it leaks water at a rate of $\frac{1}{3}$ ft^3/s. Calculate the total work performed in raising the container. The density of water is 64.2 lb/ft^3.

44. Let W be the work (against the sun's gravitational force) required to transport an 80-kg person from Earth to Mars when the two planets are aligned with the sun at their minimal distance of 55.7×10^6 km. Use Newton's Universal Law of Gravity (see Exercises 32–34 in Section 6.5) to express W as an integral and evaluate it. The sun has mass $M_s = 1.99 \times 10^{30}$ kg, and the distance from the sun to the earth is 149.6×10^6 km.