Mathematics of Motion III
The Motion of Planets

History of science

Copernicus . . . The planets revolve around the sun.

Galileo . . . Observations by telescope prove that the Copernican theory is correct.

Kepler . . . Three laws of orbits:
1. The orbits are ellipses with the sun at one focus.
2. The radial vector sweeps out equal areas in equal times.
3. $T^2 \propto a^3$.

Newton . . . There are *mathematical theories* that explain why the planets move as they do.
(laws of motion, universal gravity, calculus)

Table of variables

R	radius for a circular orbit
a	semimajor axis of an ellipse
T	period of revolution
m	mass of a planet or satellite
M	mass of the sun
G	Newton's gravitational constant

The period of revolution

This calculation is for the case of a circular orbit:

• Newton's second law says $F = ma$.

• The centripetal acceleration in $a = v^2/R$ where R is the radius of the orbit and v is the speed ($=2\pi R/T$).

• The gravitational force is $F = GMm/R^2$.

• Combine these equations $\Longrightarrow$

$$\frac{m(2\pi R/T)^2}{R} = \frac{GMm}{R^2};$$

and solve for the period of revolution T,

$$T = \sqrt{\frac{4\pi^2 R^3}{GM}}.$$

For the case of an elliptical orbit,

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}}$$

where a is the semimajor axis.

Ellipse geometry

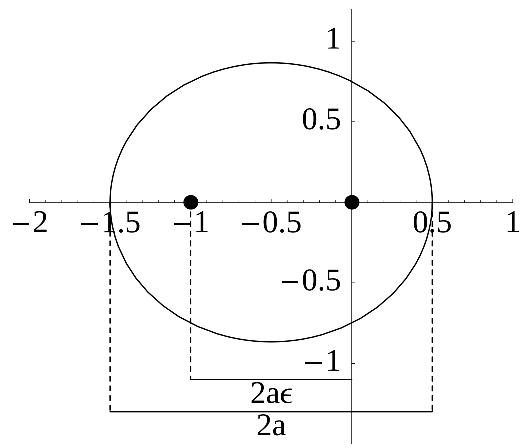
An ellipse may be specified by two fixed parameters,

$$a = \text{semimajor axis,}$$
$$\varepsilon = \text{eccentricity.}$$

The large diameter is $2a$.

The distance between the foci is $2a\varepsilon$.

Note that ε is dimensionless, and must be between 0 and 1. An ellipse with $\varepsilon = 0$ is a circle.



Units

1 AU = 1 astronomical unit = mean distance from sun to Earth = 1.496×10^{11} m.

1 y = 1 year = period of revolution of Earth = 3.15×10^7 s.

Exercises

Exercise 1

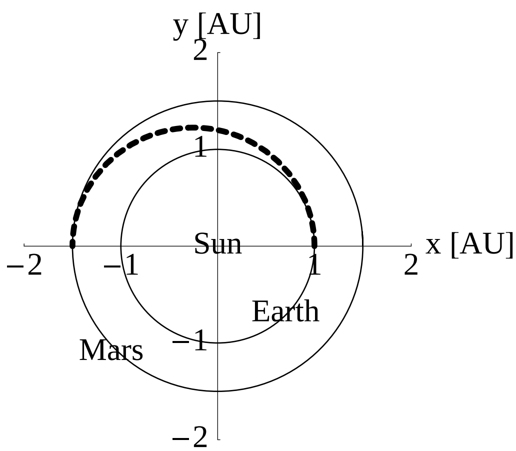
The Earth's orbit is nearly circular, with radius $R_\oplus = 1.496 \times 10^{11}$ m. From this, and the *laboratory* measurement of Newton's gravitational constant,

$$G = 6.67 \times 10^{-11} \mathrm{m}^3 \mathrm{s}^{-2} \mathrm{kg}^{-1},$$

calculate the mass of the sun.

Exercise 2 – Flight to Mars

To send a satellite from Earth to Mars, a rocket must accelerate the satellite until it is in an elliptical orbit around the sun. The satellite does not travel to Mars under rocket power, because there isn't enough fuel. It just moves in the Keplerian orbit under the influence of the sun's gravity.



The satellite orbit must have perihelion $r_- = R_E$ (=radius of Earth's orbit) and aphelion $r_+ = R_M$ (=radius of Mars's orbit) as shown in the figure. The planetary orbit radii are

$$\begin{aligned} R_E &= 1.496 \times 10^{11}\,\mathrm{m}, \\ R_M &= 2.280 \times 10^{11}\,\mathrm{m}. \end{aligned}$$

(a) What is the semimajor axis of the satellite's orbit?

(b) Calculate the time for the satellite's journey. Express the result in months and days, counting one month as 30 days.

Exercise 3 – Parametric equations for a planetary orbit

The sun is at the origin and the plane of the orbit has coordinates x and y. We can write parametric equations for the time t, and coordinates x and y, in terms of an independent variable ψ:

$$t = \frac{T}{2\pi}(\psi - \varepsilon \sin\psi) \quad (1)$$

$$x = a(\cos\psi - \varepsilon) \quad (2)$$

$$y = a\sqrt{1-\varepsilon^2}\sin\psi \quad (3)$$

The fixed parameters T, a, and ε are

T = period of revolution
a = semimajor axis
ε = eccentricity

(a) The orbit parameters of Halley's comet are

$$a = 17.9\,\mathrm{AU} \quad \text{and} \quad \varepsilon = 0.97.$$

Use Mathematica to make a parametric plot of the orbit of Halley's comet. (You only need the parametric equations for x and y, letting the variable ψ go from 0 to 2π for one revolution.)

(b) Calculate the perihelion distance. Express the result in AU.

(c) Calculate the aphelion distance. Express the result in AU. How does this compare to the radius of the orbit of Saturn, or Neptune?

(d) Calculate the period of revolution. Express the result in years.

Exercise 4

For Halley's comet, make a plot of radial distance r as a function of time t. (Note: $r = \sqrt{x^2 + y^2}$.)

Exercise 5 – Parametric surfaces

A parametric *curve* is a curve on a plane. The curve is specified by giving coordinates x and y as functions of an independent parameter t.

A parametric *surface* is a surface in 3 dimensions. The surface is specified by giving coordinates x, y, and z as functions of 2 independent parameters u and v. That is, the parametric equations for a surface have the form

$$x = f(u,v) \quad , \quad y = g(u,v) \quad , \quad z = h(u,v). \tag{4}$$

As u and v vary over their domains, the points (x, y, z) cover the surface.

The Mathematica command for plotting a parametric surface is `ParametricPlot3D`. To make a graph of the surface specified by (4), execute the command

```
ParametricPlot3D[{f[u,v],g[u,v],h[u,v]},
   {u,u1,u2},{v,v1,v2}]
```

In this command, (u_1, u_2) is the domain of u and (v_1, v_2) is the domain of v. Before giving this command you must define in Mathematica the functions `f[u,v], g[u,v], h[u,v]`. For example, for exercise (a) below you would define

```
f[u_,v_]:=Sin[u]Cos[v]

et cetera
```

Make plots of the following parametric surfaces. Hand in either the Mathematica plots, or hand drawn sketches of the surfaces. Also, in each case state in words what surface it is.

(a) For $0 \le u \le \pi$ and $0 \le v \le 2\pi$,

$$\begin{aligned} f(u,v) &= \sin u \cos v \\ g(u,v) &= \sin u \sin v \\ h(u,v) &= \cos u \end{aligned}$$

(b) For $0 \le u \le 2\pi$ and $-0.3 \le v \le 0.3$,

$$\begin{aligned} f(u,v) &= \cos u + v\cos(u/2)\cos u \\ g(u,v) &= \sin u + v\cos(u/2)\sin u \\ h(u,v) &= v\sin(u/2) \end{aligned}$$

(c) For $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$,

$$\begin{aligned} f(u,v) &= 0.2(1 - v/(2\pi))\cos(2v)(1+\cos u) + 0.1\cos(2v) \\ g(u,v) &= 0.2(1 - v/(2\pi))\sin(2v)(1+\cos u) + 0.1\sin(2v) \\ h(u,v) &= 0.2(1 - v/(2\pi))\sin u + v/(2\pi) \end{aligned}$$

Exercise 6

The Newtonian theory of motion and gravity is very accurate, but not exact. A more precise theory was developed by Albert Einstein. It is called the theory of general relativity. In Einstein's theory the orbit of a planet must be a *precessing ellipse* rather than a perfect closed ellipse. Einstein's theory agrees with planetary observations.

A mathematical representation of a precessing ellipse may be given by the parametric equations

$$\begin{aligned} x_{\rm p} &= x\cos(0.02\psi) - y\sin(0.02\psi) \\ y_{\rm p} &= x\sin(0.02\psi) + y\cos(0.02\psi) \end{aligned}$$

where $x_{\rm p}$ and $y_{\rm p}$ are the points on the precessing ellipse, and x and y are the points on the nonprecessing ellipse (given by equations (2) and (3)); ψ is the same parameter as in (2) and (3).

Hand in a plot of the precessing ellipse if the semimajor axis is $a = 1$ and the eccentricity is $\varepsilon = 0.5$.

Homework problem
due Thursday, Nov 15

Look up, e.g., in an encyclopedia or on the Internet, the semimajor axis (a) and period of revolution (T) for all nine of the planets.

(a) Record the data in the table below. For each planet express the semimajor axis in AU, and the period in y, and calculate T^2/a^3 in y^2/AU^3.

planet	a [AU]	T [y]	T^2/a^3 [y^2/AU^3]
Mercury			
Venus			
Earth			
Mars			
Jupiter			
Saturn			
Uranus			
Neptune			
Pluto			

(b) What do you notice about the values in the fourth column?

(c) Explain why the values in the fourth column are constant.

Answers Nov 8, 2001

Exercise 1

The mass of the sun is ...

Exercise 2 – Flight to Mars

(a) The semimajor axis of the satellite's orbit is ...

(b) The time for the satellite's journey is ...

Exercise 3 – Halley's comet

(a) Hand in a fraph of the orbit.

(b) The perihelion distance is ...

(c) The aphelion distance is ...

(d) The period of revolution is ...

Exercise 4

Hand in a graph of radial distance versus time.

Exercise 5 – Parametric Surfaces

Hand in plots of the surfaces; also, answer in words:

(a) What is the surface?

(b) What is the surface?

(c) What is the surface?

Exercise 6

Hand in a plot of the precessing ellipse.