MAT225 final: Fall 2005

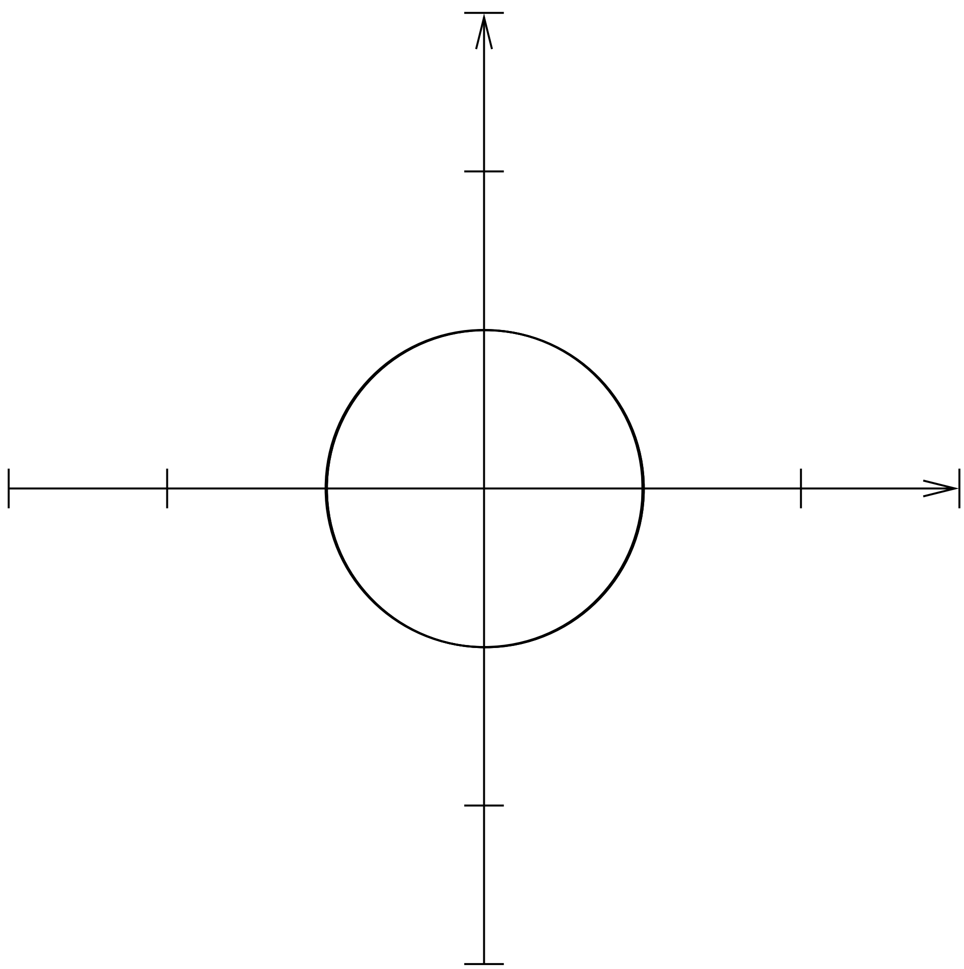
Name:

Directions: Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1 (10 pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The eigenvalues of this matrix are 3 and -1. Given the unit circle (of radius 1) in the coordinate system below, show the image of the unit circle under the transformation $A : \mathbf{x} \longrightarrow A\mathbf{x}$ in the same figure.



How much area is contained within the image?

Problem 2 (10 pts) You suspect that a linear relationship exists between homework grades and test grades, and have collected the following data: Find the best fitting line $T = mH + b$, where T

Table 1:

hw	test
.5	61
1.5	79
2.5	84
3.5	92

is the test grade, and H is the homework grade, using least squares.

1. Write the normal equations for this case.

2. Solve them for the best choice of m and b.

3. What test score would you expect for someone with a homework average of 3?

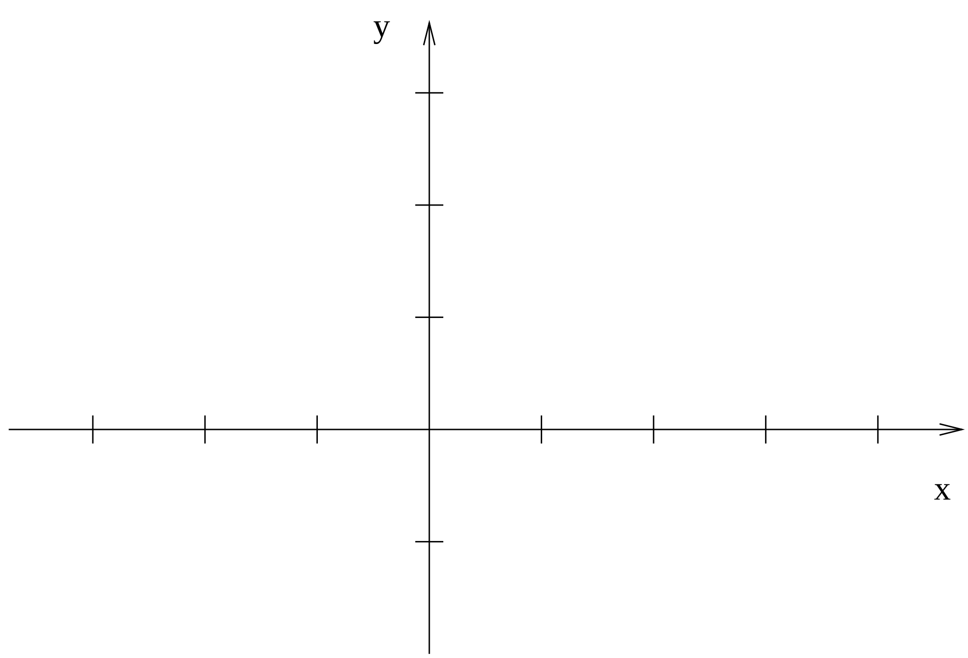
Problem 3 (10 pts) Consider the orthogonal matrix

$$U = \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{bmatrix}$$

1. Without calculation, how can we determine U^{-1}? (What is it?)

2. Without calculation, how can we determine the determinant of U? (What is it?)

3. Draw a nose in the axis below, and show its image under the transformation $U : \mathbf{x} \to U\mathbf{x}$ on the nose.



Problem 4 (10 pts) Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, where

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$$

1. What is the dimension of W?

2. Provide a basis for the orthogonal complement of W, $W^{\perp}$.

3. Write the standard basis vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as the sum of a vector in W and a vector in $W^{\perp}$.

Problem 5 (10 pts) Orthogonally diagonalize the matrix $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$.

Problem 6 (10 pts) Use row reduction to find the inverse of the matrix A, where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

Work by hand, using partial pivoting, and showing each step.

Problem 7 (10 pts). Consider the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} -4 & -3 \\ -2 & -1 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix}$$

Are these matrices linearly independent in the space $M_{2 \text{ x } 2}$ of 2 x 2 matrices with real entries?

Give the dimension and a basis for $M_{2 \text{ x } 2}$.

Problem 8 (10 pts). Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

1. Without doing any computations, how many real eigenvalues does A have (including multiplicity)? How do you know?

2. Find a basis for the column space of A, Col A.

3. Find a basis for the row space of A, Row A.

4. Find a basis for the null space of A, Nul A. [Name one eigenvalue, and its multiplicity!]

Problem 9 (10 pts). Consider

$$A = \begin{bmatrix} 4 & -3 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}.$$

1. Find the LU decomposition of A.

2. Use the LU decomposition to find the solution to the system $A\mathbf{x} = \begin{bmatrix} 9 \\ 3 \\ 4 \end{bmatrix}$

Problem 10 (10 pts). Express the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 3 & 0 & -3 \\ 4 & 0 & -4 \end{bmatrix}$$

as an outer-product.

1. What is the dimension of the column space?

2. What is the dimension of the row space?

3. What is the dimension of the null space?

4. What is the dimension of the null space of the transpose A^T?