MAT225 Test 2: Chapters 3, 4, 5

Name:

Directions: Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1 (10 pts) Verify that

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A. Explain your solution (a calculator solution alone won't amount to much...).

Problem 2 (10 pts). There are those in the world who are ignorant of linear algebra, those who are learning, and those who are proficient. The proficient lose their proficiency at a rate of 10% per year (becoming learners again), but their ranks are swollen by 40% of those learning. Those learning are recruited from the ignorant group at a rate of 1% per year, continue as learners at the rate of 30%, and the rest (not otherwise allocated) revert to the ignorant group. No one dies!

1. Draw a diagram of the three populations I, L, and P, and the flows (as proportions) between the populations. Remember that everyone goes somewhere at each time step, either back into their own group or on to a new group – for better or worse!

2. Write the matrix A that transforms the vector $\begin{bmatrix} I \\ L \\ P \end{bmatrix}$ from year to year.

3. One eigenvalue of the matrix A is 1. Find a corresponding eigenvector with positive integer components, and interpret it.

Problem 3 (10 pts). Let $B = \{1, x, x^2, x^3\}$ be the standard basis of the vector space $\mathbb{P}_3$. The Taylor series polynomial of degree three for a function f at a point a is given by

$$C(x) = f(a) + f^{'}(a)(x-a) + f^{''}(a)\frac{(x-a)^2}{2} + f^{'''}(a)\frac{(x-a)^3}{6}$$

1. What are the coefficients of the vector C in the basis $T = \{1, x-a, \frac{(x-a)^2}{2}, \frac{(x-a)^3}{6}\}$?

2. What are the coefficients of the vector C in the standard basis B?

3. Write a matrix A which allows one to transform between the two bases, and demonstrate that it works with the two vectors obtained above.

Problem 4 (10 pts). Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ -2 & 3 & 5 & -2 & -4 \\ 3 & 1 & -7 & 1 & 3 \\ 2 & 6 & -2 & 2 & 4 \end{bmatrix}$$

1. What is the rank of A?

2. Find a basis for the column space of A, Col A.

3. Find a basis for the row space of A, Row A.

4. Find a basis for the null space of A, Nul A.

Problem 5 (10 pts). Consider the vector space $M_{3 \text{ x } 3}$ of 3 x 3 matrices with real entries. Determine whether the following are subspaces:

1. Diagonal 3 x 3 matrices with real entries.

2. Scalar multiples of the 3 x 3 identity matrix.

3. Matrices of the form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

   with $a + b + c = 1$ with real entries.

4. Given fixed 3 x 3 matrix F. All matrices A with real entries such that $AF = 0_{3 \text{ x } 3}$.

Problem 6 (10 pts). Mark the following true ("T") or false ("F"). If false, indicate how to fix the statement to make it true.

1. ( ) If λ is an eigenvalue of matrix A, then the determinant of matrix $A - \lambda I$ is zero.

2. ( ) All invertible matrices are diagonalizable.

3. ( ) If a vector space V has n linearly independent vectors, then the dimension of V is greater than n.

4. ( ) $\mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$ in general.

5. ( ) If $A = PBP^{-1}$, then A and B have the same eigenvalues.

Extra Credit (5 pts). Is it possible for a 3 x 3 matrix to have two real eigenvalues and one complex eigenvalue? Explain!