MAT225 final: Fall 2004

Name:

Directions: Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Part I: New Stuff

Problem 1 (10 pts) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \\ -2 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 21 \\ 1 \end{bmatrix}$$

Problem 2 (10 pts) Consider the problem of predicting final exam grades based on age, height, weight, and gender (indicated by a -1 for men, 1 for women):

$$grade = x_0 + x_1\ age + x_2\ height + x_3\ weight + x_4\ gender$$

1. If we were to write this as a matrix system of the form $A\mathbf{x} = \mathbf{b}$, describe A and the vector $\mathbf{b}$.

2. In a class of 23 students, what is the chance that $\mathbf{b}$ is in the column space of A? Why?

3. What's your strategy for finding the weights x_i?

Problem 3 (10 pts) Consider the orthogonal matrix U. What can you say about

1. U^{-1}?

2. the determinant of U?

3. the effect of the transformation $U : \mathbf{x} \rightarrow U\mathbf{x}$? (Describe the geometry.)

4. $||U\mathbf{x}||$?

Problem 4 (10 pts) Let W be the subspace spanned by the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Write

$$y = \begin{bmatrix} -4 \\ -21 \\ -1 \end{bmatrix}$$

as the sum of a vector in W and a vector orthogonal to W.

Problem 5 (10 pts) Orthogonally diagonalize the matrix $A = \begin{bmatrix} 9 & -3 \\ -3 & 17 \end{bmatrix}$.

Problem 6 (10 pts) Compute the SVD of the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$. Write it in reduced form, and in the full form (in which U is 3 x 3).

Part II: Old Stuff

Problem 7 (10 pts) Use row reduction to solve the equation $A\mathbf{x} = \mathbf{b}$, represented by the augmented system

$$\begin{bmatrix} 4 & -4 & -4 & 4 & 12 \\ -1 & 3 & 7 & -1 & 3 \\ -2 & 6 & 14 & 6 & 46 \end{bmatrix}.$$

Work by hand, using partial pivoting, to write the system in reduced row echelon form (showing each step).

Problem 8 (10 pts). Consider the functions $f_1(x) = 3x^2 - 2x + 1$, $f_2(x) = x^2 + 2x + 4$, and $f_3(x) = -x^2 + 6x + 7$, in the space $\mathbb{P}_3$. Are these functions linearly independent?

Problem 9 (10 pts). Describe the geometry of the solution set of $A\mathbf{x} = \mathbf{b}$ when A is 3 x 3.

1. Consider the cases where the rank of A ranges from 0 to 3.

2. When A is rank 3, what does the condition number tell you?

3. What can you say about the determinant of A, depending on the rank of A?

Problem 10 (10 pts). Consider the matrix

$$A = \begin{bmatrix} 2 & 5 & -8 & 0 & 17 \\ -2 & -8 & 14 & -6 & 4 \\ 1 & 6 & -11 & 7 & -16 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

1. What is the rank of A?

2. Find a basis for the column space of A, Col A.

3. Find a basis for the row space of A, Row A.

4. Find a basis for the null space of A, Nul A.

Problem 11 (10 pts). Consider

$$A = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}$$

1. Calculate the inverse of the matrix A.

2. Check that $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

3. Use the LU decomposition to find the solution to the system $A\mathbf{x} = \begin{bmatrix} -10 \\ -10 \end{bmatrix}$

Problem 12 (10 pts). An ugly little intestinal worm is afflicting you, whose life cycle can be modelled as three stages: juvenile, youngsters, and adults. Each day, the populations cycle, as follows:

- Ten percent of the juveniles "graduate" to youngster status, while the rest die. New juveniles are produced by adults (twenty percent of the adults give birth to a new juvenile).
- Thirty percent of the youngsters remain youngsters; ten percent die; sixty percent graduate to adulthood.
- Eighty percent of the adults remain adults; the others are excreted from the system.

1. Draw a diagram of the lifecycle flows of the populations J, Y, and A, and write the matrix T that transforms the vector $\begin{bmatrix} J \\ Y \\ A \end{bmatrix}$ from day to day.

2. One eigenvalue of the matrix T is .2. What does the long term future of the worm look like? (More importantly, how does your future look?!)