Euler E521. From the Memoirs of the Royal Academy of Sciences.

Excerpts from several letters of M. Euler to M. the Marquis de Condorcet

November 1775

The integral of the formula, $\frac{x^m - x^n}{lx} \cdot \frac{dx}{n}$, taken from $x = 0$ to $x = 1$, is $l\frac{m}{n}$.

The integral of the formula $\frac{x^{m-1}\partial x}{(1+x^n)lx}$, taken from $x = 0$ to $x = \infty$ is $l \tan \frac{m\pi}{n}$, where π denotes the angle of 180 degrees.

February 2,1776

Demonstration of the Two Proceeding Theorems

Let Q be an unknown function of the two variables x & y, and look for the quantity Z, such that $\left(\frac{\partial\partial Z}{\partial x \partial y}\right) = Q$. Make a double integration, the first where x alone is taken as a variable, and the other where only y varies. The first will be extended from $x = 0$ to $x = 1$, & the other from $y = 0$ to $y = n$. By the nature of such formulas, we will have, in a double manner, either $Z = \int \partial x \int Q\partial y$, or $Z = \int \partial y \int Q\partial x$. Now suppose that $Q = x^y$, then $\int Q\partial y = \frac{x^y}{lx} - \frac{1}{lx}$. Notice that this integral vanishes when $y = 0$. Now let $y = n$ and we will have $\int Q\partial y = \frac{x^n - 1}{lx}$, leaving $Z = \int \frac{(x^n-1)\partial x}{lx}$. Next we have $\int Q\partial x = \frac{x^{y+1}}{y+1}$, which vanishes when $x = 0$, substituting $x = 1$, this results in $\int Q\partial x = \frac{1}{y+1}$, and thus $Z = \int \frac{\partial y}{y+1} = l(y + 1)$, (an expression which disappears when $x = 0$.) Substitute $y = n$, and we obtain $Z = l(n + 1)$, and by consequence it is certain that the integral $\int \frac{\partial x(x^n-1)}{lx}$, taken from $x = 0$ to $x = 1$, is $l(n + 1)$.

For the other, more complicated, integral that I wrote to you about, I have supposed that $Q = \frac{x^{m-\gamma} - x^{m+\gamma}}{(1+x^{am})x}$. From this, taking x as a constant, and because $\int x^{m-y}\partial y = -\frac{x^{m-y}}{lx}$, and also $\int x^{m+y}\partial y = -\frac{x^{m+y}}{lx}$, we will have $\int Q\partial y = \frac{x^{m+n} - x^{m-n}}{(1+x^{am})xlx}$, leaving us $Z = \int \frac{(x^{m+n} - x^{m-n})\partial x}{(1+x^{am})xlx}$.

The other integration gives $\int Q\partial x = \int \frac{(x^{m-n} - x^{m+n})\partial x}{(1+x^{am})x}$, where the integral is taken from $x = 0$ to $x = 1$. But I have shown elsewhere that this integral reduces to the form $\frac{\pi}{am \cos \frac{\pi y}{am}}$, from which we obtain $Z = \int \frac{\pi \partial y}{am \cos \frac{\pi y}{am}}$. For this integral, let $\frac{\pi y}{am} = \varphi$ to give $Z = \int \frac{\partial\varphi}{\cos\varphi} = \int \frac{\partial\varphi}{\sin(90^d + \varphi)}$. The integral of this is $l \tan(45^d + \frac{1}{2}\phi)$, which leaves $Z = l \tan(45^d + \frac{\pi y}{4m})$, which vanishes when $y = 0$. Making $y = n$, we get $Z = l \tan(45^d + \frac{\pi n}{4m})$, from which it is clear that under the present conditions we will have $\int \frac{(x^{m+n-1} - x^{m-n-1})\partial x}{(1+x^{am})lx} \left\{ \begin{matrix} \text{from} & \text{x=0} \\ \text{to} & \text{x=1} \end{matrix} \right\} = l \tan(45^d + \frac{\pi n}{4m})$.

By these two examples it can be easily seen that this procedure merits the full intention of Geometers. The idea which first led me to this research followed an entirely different approach, which follows. I considered the formula $\int \frac{(x-1)\partial x}{lx}$, and in place of lx I substituted $\frac{x^\omega - 1}{\omega}$, with ω infinitely small - or equivalently $lx = i(x^{1/i} - 1)$, with i an infinitely large number. Then allowing $x^{1/i} = z$, or, equivalently, $x = z^i$, the endpoints of integration which were $x = 0$ to $x = 1$, become $z = 0$ to $z = 1$. When these values are substituted the integrand becomes $\frac{(z^i-1)z^{i-1}\partial z}{z-1}$. Now the fraction $\frac{(z^i-1)}{z-1}$, or equivalently, $\frac{(1-z^i)}{1-z}$, can be replaced by the series $1 + z + z^2 + z^3 + \cdots + z^{i-1}$. When this is multiplied and integrated it gives

$$\frac{z}{i} + \frac{z^{i+1}}{i+1} + \frac{z^{i+2}}{i+2} + \frac{z^{i+3}}{i+3} + \cdots + \frac{z^{2i-1}}{2i-1}.$$

Now setting $z = 1$, the value will become

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \cdots + \frac{1}{2i-1}.$$

The value of this series is $l2$, from which it follows that $\int \frac{(x-1)\partial x}{lx} \left\{ \begin{array}{cc} \text{from} & \text{x=0} \\ \text{to} & \text{x=1} \end{array} \right\}$ is $l2$.

In order show that the sum of the series which we will call A, one needs only to notice that,

$$A = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots +$$
$$\cdots + \frac{1}{i-1} + \frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \cdots$$
$$\cdots + \frac{1}{2i-1} - (1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots + \frac{1}{i-1}).$$

Now since the first series has twice the number of terms as the second, we can subtract each term of the second series from every other term of the first, and we will have:

$$A = 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + \cdots +$$
$$\cdots + \frac{1}{i-1} + \frac{1}{i} + \frac{1}{i+1} + \frac{1}{2i-1}$$
$$-1 - 1/2 - 1/3 - 1/4 - 1/5 + \cdots,$$

or

$$A = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \cdots = l2.$$

Another Theorem

Allowing the letters $\alpha, \beta, \gamma, \delta$, etc. to denote the coefficients of a binomial raised to an exponent n, that is

$$(1+x)^n = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + etc,$$

then

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdots \frac{4n-1}{n}.$$

For example, when n=6, then $\alpha = 6, \beta = 15, \gamma = 20, \delta = 15, \epsilon = 6, \zeta = 1$, and all the rest are zero. Thus we will have

$$1 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdot \frac{22}{6}.$$

A direct proof of this result seems to be extremely difficult.

Proof of the Theorem

September 1776

Let

$$(1+z)^n = 1 + \binom{n}{1} z + \binom{n}{2} z^2 + \binom{n}{3} z^3 + \&c,$$

from which we see that $\binom{n}{0} = 1$, as does $\binom{n}{n}$. It also follows that $\binom{n}{p} z = \binom{n}{n-p}$, and it is clear that the value of the expression $\binom{n}{p}$ is always zero both in the case that p is a negative number and when p is larger than n, which extends the notation to all numbers. Finally, we see that the value of $\binom{n}{p}$ is $= \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdots \frac{n-p+1}{p}$.

This done, we will look at the coefficients of the next power $(1+z)^{n+1}$. We have that $\binom{n+1}{p+1} = \binom{n}{p} + \binom{n}{p+1}$. In a like manner we also have, $\binom{n}{p+1} + \binom{n}{p+2} = \binom{n+1}{p+2}$. Adding these two equation we get

$$\binom{n}{p} + 2\binom{n}{p+1} + \binom{n}{p+2} = \binom{n+1}{p+1} + \binom{n+1}{p+2} = \binom{n+2}{p+2}.$$

In the same way we also have

$$\binom{n}{p+1} + 2\binom{n}{p+2} + \binom{n}{p+3} = \binom{n+2}{p+3},$$

and when this equation is added to the previous we obtain,

$$\binom{n}{p} + 3\binom{n}{p+1} + 3\binom{n}{p+2} + \binom{n}{p+3} = \binom{n+2}{p+2} + \binom{n+2}{p+3} = \binom{n+3}{p+3}.$$

Likewise we have

$$\left(\frac{n}{p+1}\right)+3\left(\frac{n}{p+2}\right)+3\left(\frac{n}{p+3}\right)+\left(\frac{n}{p+4}\right)=\left(\frac{n+3}{p+4}\right),$$

which when added to the previous equation gives us,

$$\left(\frac{n}{p}\right)+4\left(\frac{n}{p+1}\right)+6\left(\frac{n}{p+2}\right)+4\left(\frac{n}{p+3}\right)+\left(\frac{n}{p+4}\right)=\left(\frac{n+3}{p+3}\right)+\left(\frac{n+3}{p+4}\right)=\left(\frac{n+4}{p+4}\right).$$

From this it is easy to conclude that in general it is true that

$$\left(\frac{n}{p}\right)+\left(\frac{m}{1}\right)\left(\frac{n}{p+1}\right)+\left(\frac{m}{2}\right)\left(\frac{n}{p+2}\right)+\left(\frac{m}{3}\right)\left(\frac{n}{p+3}\right)+etc=\left(\frac{n+m}{p+m}\right).$$

Here we have a regular progression, in which each term is the product of two coefficients of different powers of the binomial, where the general term can be given by the formula $\left(\frac{m}{x}\right)\cdot\left(\frac{n}{p+x}\right)$. If we substitute in for x the numbers 0,1,2,3,4, & etc until one of the coefficients vanishes, the sum of the progression will always be $\left(\frac{n+m}{p+m}\right)=\left(\frac{n+m}{n-p}\right)$.

It is from this that the theorem I communicated to you can be proven. Take $m=n$ and $p=0$, and we will use one of the infinitely many cases of the summation that I have shown you here. In this case we will have the summation,

$$1^2+\left(\frac{n}{1}\right)^2+\left(\frac{n}{2}\right)^2+\left(\frac{n}{3}\right)^3+\&c.=\left(\frac{2n}{n}\right).$$

When the right side is expanded we get

$$\frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{4}\cdots\frac{n+1}{n}\cdot.$$

This, it is easy to show, is equal to,

$$\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\cdots\frac{4n-2}{n}.$$

It is truly remarkable that this summation has also been given, even when the exponents m and n are arbitrary fractions, by the way of interpolation. We can assign the correct value of $\left(\frac{m+n}{m+p}\right)$ and if the value has not been assigned in this case it can be recovered by integrals. Let $l\frac{1}{x}=u$, then

$$\left(\frac{m+n}{m+p}\right)=\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x\int u^{n-p}\partial x}\left\{\begin{array}{cc}\text{from} & \text{x=0}\\ \text{to} & \text{x=1}\end{array}\right\}$$

If λ is any positive number, then $\int u^\lambda\partial x=1\cdot2\cdot3\cdot4\cdots\lambda$, and from that we have, $\int u^{\lambda+1}\partial x=(\lambda+1)\int u^\lambda\partial x$, and $\int u^{\lambda+2}\partial x=(\lambda+1)(\lambda+2)\int u^\lambda\partial x$, and etc. This result will be true, whatever number is substitutd for λ. Now taking $\lambda=-\frac{1}{2}$, I have shown elsewhere that $\int\frac{\partial x}{\sqrt{u}}=\pi$, and $\int\partial x\sqrt{u}=\frac{1}{2}\sqrt{\pi}$, where π

denotes the circumference of a circle with diameter =1. Now if $m = n = \frac{1}{2}$ and $p = 0$, then the coeffeicients of $(1+x)^{1/2}$ are

$$1 + \frac{1}{2} - \frac{1\cdot 1}{2\cdot 4} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + \&c.$$

We then have the series of squares,

$$1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 1}{2\cdot 4}\right)^2 + \left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2 + \&c,$$

of which the sum will be $\frac{\int u\partial x}{\int \partial x\sqrt{u}\int \partial x\sqrt{u}} = \frac{4}{\pi}$, because $\int u\partial x = 1 \& \int \partial x\sqrt{u} = \frac{1}{2}\sqrt{\pi}$. This agrees perfectly with the sum that was obtained by the method of approximation.