Sometimes it’s desirable to say an event $U$ has no probability value assigned. The **inner** and **outer measures** associated with a single measure $\mu$ give values for all events $U$, even where $\mu$ is undefined:

- $\mu_*(U) =$ **maximum** $\mu$ measure of all $U$’s **subsets**
- $\mu^*(U) =$ **minimum** $\mu$ measure of all $U$’s **supersets**

Treat the *interval* $[\mu_*(U), \mu^*(U)]$ as a surrogate for some unavailable number “probability of $U$.”

**EXAMPLE**

A ball is drawn from an urn containing 30 red, 70 blue-or-yellow balls.

$W = \{ \emptyset, \{r\}, \{b\}, \{y\}, \{r,b\}, \{r,y\}, \{b,y\}, \{r,b,y\} \}.$

It’s meaningful to define a probability measure $\mu$ only on a subalgebra of $W$:

$\{ \emptyset, \{r\}, \{b,y\}, \{r,b,y\} \} \n\n0 \quad 0.3 \quad 0.7 \quad 1 \n
\mu_*(\{r,y\}) = \max\{\mu(\emptyset), \mu(\{r\})\} = \max\{0, 0.3\} = 0.3
\mu^*(\{r,y\}) = \min\{\mu(\{r,b,y\})\} = \min\{1\} = 1
Sometimes it’s desirable to say an event $U$ has many probability values assigned. The lower and upper probabilities associated with a given set of measures $\mathcal{P} = \{\mu_1, \ldots, \mu_n\}$ give single values for all events $U$:

- $\mathcal{P}^*(U) = \text{minimum}$ of the values assigned to $U$ by measures in $\mathcal{P}$
- $\mathcal{P}^*(U) = \text{maximum}$ of the values assigned to $U$ by measures in $\mathcal{P}$

Treat the interval $[\mathcal{P}^*(U), \mathcal{P}^*(U)]$ as a surrogate for some unavailable number “probability of $U$.”

**EXAMPLE**

A ball is drawn from an urn containing 30 red, 70 blue-or-yellow balls. Let’s define a family of $\mathcal{P} = \{\mu_0, \ldots, \mu_{70}\}$ probability measures on these events.

$$W = \{ \emptyset, \{r\}, \{b\}, \{y\}, \{r,b\}, \{r,y\}, \{b,y\}, \{r,b,y\} \}.$$  

$$\mu_i : 0 \quad 0.3 \quad i/100 \quad 0.7 - i/100 \quad 0.3 + i/100 \quad 1 - i/100 \quad 0.7 \quad 1$$

$$\mathcal{P}^*(\{r,y\}) = \min \{\mu_i(\{r,y\}) \mid i=0\ldots70\} = \min \{1 - i/100 \mid i=0\ldots70\} = 0.3$$

$$\mathcal{P}^*(\{r,y\}) = \max \{\mu_i(\{r,y\}) \mid i=0\ldots70\} = \max \{1 - i/100 \mid i=0\ldots70\} = 1$$
\[ \mathcal{O}_\mu : \text{all possible extensions of } \mu \text{ to the full algebra of all subsets of } \mathbb{W} \]

defined on a subalgebra of \( \mathbb{W} \)

Theorem:
This measure is the same as the lower probability measure from the set \( \mathcal{O}_\mu \).

So: Any inner measure results from a lower probability. [But not conversely.]
(To get an inner measure \( \mu_* \), form the set of all of \( \mu \)'s extensions and take the lower prob.)